Selected topics in Algebra - Homological methods in comm. algebra

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Overview of the course:

- Tor and Ext of modules over a ring:

For a short exact sequence of R-modules,
$$0 \to M' \to M \to M'' \to 0,$$
we have the exact sequences

- $M' \otimes_R T \to M \otimes_R T \to M'' \otimes_R T \to 0$
- $0 \to \mathrm{Hom}_R(T, M') \to \mathrm{Hom}_R(T, M) \to \mathrm{Hom}_R(T, M'')$
- $0 \to \mathrm{Hom}_R(M'', T) \to \mathrm{Hom}_R(M, T) \to \mathrm{Hom}_R(M', T)$

which typically don't extend to short exact sequences.
For example, for $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$, the sequence
$$\begin{array}{ccccccccc} 0 \to & \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} & \xrightarrow{\cdot 2} & \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} & \to & \mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} & \to 0 \\ \| & \| & & \| & & \| & \\ 0 \to & \mathbb{Z}/2\mathbb{Z} & \xrightarrow{2} & \mathbb{Z}/2\mathbb{Z} & \xrightarrow{\mathrm{id}} & \mathbb{Z}/2\mathbb{Z} & \to 0, \end{array}$$
is not exact on the left.
Also, as $\mathrm{id}_{\mathbb{Z}/2\mathbb{Z}} : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ cannot be lifted to a map $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$, the sequence
$$0 \to \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \xrightarrow{\cdot 2} \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \to 0$$
is not exact on the right.
Finally, as $2\mathbb{Z} \to \mathbb{Z} : n \mapsto \frac{n}{2}$ cannot be extended to $\mathbb{Z}$, the sequence
$$0 \to \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \xrightarrow{\cdot 2} \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \to 0$$
is not exact on the right.
To solve this problem, one constructs derived functors Tor and Ext. These give long exact sequences
$$\to \mathrm{Tor}_1^R(T, M') \to \mathrm{Tor}_1^R(T, M) \to \mathrm{Tor}_1^R(T, M'') \to M' \otimes_R T \to M \otimes_R T \to M'' \otimes_R T \to 0$$
$$0 \to \mathrm{Hom}_R(T, M') \to \mathrm{Hom}(T, M) \to \mathrm{Hom}(T, M'') \to \mathrm{Ext}_R^1(T, M) \to$$
$$0 \to \mathrm{Hom}_R(M'', T) \to \mathrm{Hom}(M, T) \to \mathrm{Hom}(M', T) \to \mathrm{Ext}_R^1(M'', T) \to$$
measuring the non-exactness of the first sequences.

▷ Serre's characterization of regularity

For a Noetherian local ring $(R, \mathfrak{m})$, the following are equivalent:

a) $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = \dim(R)$ (for $k = R/\mathfrak{m}$)

b) There is some vanishing bound for $\mathrm{Tor}_*^R(\cdot,\cdot)$

$\tilde{b}$) $\dim(R)$ is a — " — — " — — " —

c) There is some vanishing bound for $\mathrm{Ext}_R^*(\cdot,\cdot)$

$\tilde{c}$) $\dim(R)$ is a — " — — " — — " —

Corollary: If R is a regular local ring and $\mathfrak{p} \subset \mathrm{Spec}\, R$ is a prime ideal of R, then $R_\mathfrak{p}$ is local regular.

Theorem: If the Noetherian ring R is regular or Cohen-Macauley, then it is universally catenary:

If $X = \mathrm{Spec}\, A$, where A is an R-algebra of finite type and if $B \subseteq C \subseteq D$ are irreducible closed subsets of X, then

$$\mathrm{codim}(B, C) + \mathrm{codim}(C, D) = \mathrm{codim}(B, D)$$

($\leq$ always holds)

(In other words, X is catenary.)

▷ Injective and projective resolutions

- If $\mathrm{Tor}_i^R(M, -)$ vanishes for $i \geq 1$, M is a flat R-module.
- If $\mathrm{Ext}_R^i(-, M)$ vanishes for $i \geq 1$, M is an injective R-module
- If $\mathrm{Ext}_R^i(M, -)$ vanishes for $i \geq 1$, M is a projective R-module.

Proposition 1 (Baer). For an R-module N, the following are equivalent:

a) The functor $\mathrm{Hom}_R(-, N)$ is exact.

b) For any embedding $M' \to M$ of R-modules, the map $\mathrm{Hom}(M, N) \to \mathrm{Hom}(M', N)$ is surjective.

c) The previous holds for $M = R$. I.e., for $I \subseteq R$ an ideal, any morphism $I \to N$ of R-modules extends to a homomorphism $R \to N$.

Such an R-module is called injective

Remark: a) As $\mathrm{Hom}(R, M) \xrightarrow{\cong} M$, $\varphi \longmapsto \varphi(1)$, condition c) can be reformulated as the condition that any morphism $I \to N$ has the form $i \mapsto i \cdot n$ for some $n \in N$.

β) Note that c) is trivial when $I = 0$

γ) When $R = \mathbb{Z}$, c) amounts to the divisibility of the abelian group N: $N \xrightarrow{m} N$ is surjective when $m \neq 0$.

1. Tor and Ext of R-modules

1.1 Injective and projective modules. Properties of Ext^*_R

Proposition 1.1.1 (Baer's criterion). For an R-module N, the following conditions are equivalent:

a) $\mathrm{Hom}(\cdot, N)$ is an exact functor

b) If M is a submodule of M', any $f \in \mathrm{Hom}_R(M, N)$ extends to some $\varphi \in \mathrm{Hom}_R(M', N)$.

c) The same condition for $M' = R$, $I = M$ an ideal of R.

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, hook] & N \\ M' \arrow[ur, dashed, "\varphi"'] & \end{tikzcd}$$

Definition 1.1.0. Such an R-module is called *injective*

Proof of proposition 1.1.1. b) $\Rightarrow$ c) Trivial

c) $\Rightarrow$ b) Let $\mathfrak{M} = \{ (\tilde{M}, \tilde{\varphi}) \mid M \subseteq \tilde{M} \subseteq M',\ \tilde{\varphi} \in \mathrm{Hom}_R(\tilde{M}, N),\ \tilde{\varphi}|_M = f \}$

This is partially ordered by

$$(\tilde{M}, \tilde{\varphi}) \preccurlyeq (\hat{M}, \hat{\varphi}) \iff \tilde{M} \subseteq \hat{M} \text{ and } \hat{\varphi}|_{\tilde{M}} = \tilde{\varphi}$$

It is easy to see that we may apply Zorn's lemma, so $\mathfrak{M}$ has an $\preccurlyeq$-maximal element (M_*, φ_*).

Assume for contradiction that $M_* \subsetneq M'$, so there is an element $m \in M' \setminus M_*$. Let $I = \{ r \in R \mid rm \in M_* \}$, and define $g: I \to N$ by $g(r) = \varphi_*(rm)$. By c) there is a $\psi: R \to N$ extending g.

~~Let $\tilde{M} = M_* + Rm$ and $\tilde{\varphi}(m_* + rm) =$~~

This implies that there is some $v \in N$ such that

$$\varphi_*(rm) = rv$$

for all $r \in I$. Let $\tilde{M} = M_* + Rm$ and $\tilde{\varphi}(m_* + rm) := \varphi_*(m_*) + rv$.

Then $\tilde{\varphi}$ is well-defined, but $(M_*, \varphi_*) \preccurlyeq (\tilde{M}, \tilde{\varphi})$, a contradiction.

a) $\Leftrightarrow$ b) is easy, as $0 \to \mathrm{Hom}(Q, N) \to \mathrm{Hom}(X, N) \to \mathrm{Hom}(Y, N)$ is exact when $Y \to X \to Q \to 0$ is exact, and b) implies exactness at the right end when $Y \to X$ is injective.

Definition 1.1.1 If R is a domain and M is an R-module, then M is called *divisible* if $M \xrightarrow{\cdot r} M$ is surjective for all $r \in R \setminus \{0\}$.

Corollary 1.1.1 α) When R is a domain, then M is divisible if and only if c) of prop 1.1.1 holds if $I \subseteq R$ is a principal ideal.

β) Any injective module N is divisible in the following sense: if $r \in R$ is not a zero-divisor, $N \xrightarrow{\cdot r} N$ is surjective. This is equivalent to c) for $I = r \cdot R$.

γ) In particular, if N injective, $S \subseteq R$ multiplicative, then $N \to N_S$ is surjective.

Remark: Note that $R = \mathbb{Z}/p^2\mathbb{Z}$ is an injective module over itself, but $R \xrightarrow{p} R$ fails to be surjective.

Corollary 1.1.2: Every module over a PID is injective iff it is divisible.

Remark. The same works for Dedekind domains. see corollary 1.1.6.

Corollary 1.1.3 When R is a PID, then any quotient of an injective R-module is injective.

The category of R-modules has sufficiently many injective objects, in the sense that for any object X there is a monomorphism $X \to I$ with I injective. Thus every R-module X has an injective resolution, i.e an exact sequence

$$0 \to X \to I^0 \to I^1 \to I^2 \to \dots$$

In fact every R-module has an injective resolution of length 1.

$$0 \to X \to I^0 \to I^1 \to 0$$

Proof The first assertion follows, as the quotient of a divisible module is divisible.

Note that K/R is divisible, hence injective. (Where K is the quotient field of R.) If M is any R-module and $m \in M \setminus \{0\}$ we have a homomorphism from $Rm \subseteq M$ to K/R (when $I = \mathrm{Ann}_R(m)$ is non-zero, $I = r \cdot R$ with $r \neq 0$, and the morphism $\rho m \to \frac{\rho}{r} \bmod R \in K/R$) or from $R\cdot m$ to K (if $I = \mathrm{Ann}_R(m) = 0$, sending ρm to ρ.)

By injectivity of K/R and K as R-modules, there is an extension $\varphi_m : M \to \left(\begin{smallmatrix} K/R \\ \text{or} \\ K \end{smallmatrix}\right)$ of this homomorphism, satisfying $\varphi_m(m) \neq 0$. Let $I_m = K/R$ or $I_m = K$ be the target of φ_m and $I = \prod_{m \in M \setminus \{0\}} I_m$. Then I is divisible hence injective and $M \to I$ $\quad p \mapsto (\varphi_m(p))_{m \in M \setminus \{0\}}$, is a monomorphism.

As a quotient of $I^0 = I$, $I^1 = \mathrm{coker}(M \to I^0)$ is also injective, hence $0 \to M \to I^0 \to I^1 \to 0$ is an injective resolution of length 1.

Example For $R = \mathbb{Z}/p^2\mathbb{Z}$, then $M = p\mathbb{Z}/p^2\mathbb{Z}$ has as injective resolution

$$p\mathbb{Z}/p^2\mathbb{Z} \hookrightarrow R \xrightarrow{p} R \xrightarrow{p} R \longrightarrow \dots$$

Proposition 1.1.2. For any ring R, the category of R-modules has sufficiently many injective objects.

This will follow from point 2 and 3 of lemma 1.1.1

Theorem (Grothendieck) Any AB5 category with a generator has sufficiently many injective objects

Lemma 1.1.1: Let R be any ring

a) The forgetful functor R-Mod $\longrightarrow$ abelian groups ($= \mathbb{Z}$-modules) has a right adjoint functor $\mathcal{R}$ given by

(or $\varphi(\rho r)$ for left R-mod)

$$\mathcal{R}G = \operatorname{Hom}_{\mathbb{Z}}(R, G) \qquad \big((r\varphi)(\rho) := \varphi(r\rho) \big)$$

b) For an injective abelian group I, $\mathcal{R}(I)$ is an injective R-module.

c) Let M be any R-module and I an injective abelian group and $M \xrightarrow{i} I$ an injective homomorphism of abelian groups. Then the R-homomorphism $M \longrightarrow \mathcal{R}I$ obtained by applying the adjunct relation

$$\operatorname{Hom}(M, I) \cong \operatorname{Hom}_R(M, \mathcal{R}I)$$

is injective.

Proof: (a) $\mathcal{R}G$ is an R-module which depends functorially on G. We have

$$\operatorname{Hom}_{\mathbb{Z}}(M, G) \xrightarrow{\cong} \operatorname{Hom}_R(M, \mathcal{R}G)$$
$$\varphi \longmapsto (m \mapsto (r \mapsto \varphi(rm))) = \psi$$
$$\varphi = (m \mapsto \psi(m)(1)) \longleftarrow\!\shortmid \ \psi$$

Note that this ψ is indeed R-linear. We reobtain ψ from φ, since

$$(\psi(m))(r) = ((r\psi)(m))(1) = (\psi(rm))(1) = \varphi(rm)$$

b) It is a general fact that the right adjoint R of an exact functor L preserves injectivity of objects while the left adjoint L of an exact functor preserves projectivity of objects.
For instance, if L is exact and I is an injective object in $\mathcal{B}$ and $X \to Y$ is a monomorphism in $\mathcal{A}$, then the the upper arrow in the diagram

$$\begin{array}{ccc} \operatorname{Hom}_{\mathcal{A}}(Y, RI) & \longrightarrow & \operatorname{Hom}_{\mathcal{B}}(X, RI) \\ \downarrow \cong & & \downarrow \cong \\ \operatorname{Hom}_{\mathcal{B}}(LY, I) & \longrightarrow & \operatorname{Hom}_{\mathcal{B}}(LX, I) \end{array}$$

is surjective. Since the lower one is surjective, as I is injective and $LX \to LY$ is a monomorphism.

c) Let $\varphi: M \to I$ be injective, where I is an injective abelian group. The corresponding morphism $\psi: M \longrightarrow \mathcal{R}I$ sends $m \in M$ to $\psi(m): R \to G$ given by $\psi(m)(r) = \varphi(rm)$, which is non-zero for $m \neq 0$, since $\psi(m)(1) = \varphi(m) \neq 0$.

D. Excursion: Derived functors, Ext

Definition D.1: Let $\mathcal{A}$ and $\mathcal{B}$ be Abelian categories (e.g. R modules)
A homological δ-functor $F_*: \mathcal{A} \to \mathcal{B}$ is a sequence $(F_i)_{i=0}^\infty$ of additive functors $\mathcal{A} \to \mathcal{B}$ together with a ~~…~~ sequence of natural transformations

$$\delta_i : F_{i+1}(A'') \to F_i(A')$$

on the category of short exact sequences $0 \to A' \to A \to A'' \to 0$ in $\mathcal{A}$ s.t. the sequence

$$\cdots \to F_{i+1}(A'') \xrightarrow{\delta} F_i(A') \to F_i(A) \to F_i(A'') \xrightarrow{\delta} F_{i-1}(A') \to \cdots$$

$$\cdots \to F_1(A'') \to F_0(A') \to F_0(A) \to F_0(A'') \to 0$$

is exact.

A morphism $F_* \xrightarrow{\varphi} G_*$ between homological functors is a sequence $(\varphi_i)_{i=1}^\infty$ of natural transformations $\varphi_i: F_i \to G_i$ s.t. for any s.e.s. as above, there is a comm diagram

$$\begin{tikzcd}
F_{i+1}(A'') \arrow[r, "\delta"] \arrow[d, "\varphi_{i+1}"] & F_i(A') \arrow[d, "\varphi_i"] \\
G_{i+1}(A'') \arrow[r, "\delta"] & G_i(A')
\end{tikzcd}$$

The same can be done for a cohomological functor

Let $F: \mathcal{A} \to \mathcal{B}$ be right exact: for any short exact sequence $0 \to A' \to A \to A'' \to 0$, the sequence

$$FA' \to FA \to FA'' \to 0$$

is exact. A left derived functor of F is a homological functor $L_*F: \mathcal{A} \to \mathcal{B}$ with a natural isomorphism $L_0F \cong F$ such that for any homological functor $\Phi_*: \mathcal{A} \to \mathcal{B}$, any natural transformation

$$\Phi_0 \to L_0F$$

extends in a unique way to a morphism $\Phi_* \to L_*F$ of homological functors.

Similarly, F is left exact if $0 \to FA' \to FA \to FA''$ is exact for all such ses as above. A right derived functor R^*F of F is a cohomological functor ~~…~~ with a natural isomorphism $R^0F \cong F$ s.t. for any cohomological functor $\Phi^*: \mathcal{A} \to \mathcal{B}$, any natural transformation $R^0F \to \Phi^0$ extends in a unique way to $R^*F \to \Phi^*$.

<u>Remark D.1</u> a) It follows that derived functors are unique up to unique isomorphism of (co-) homological functors if they exist.

b) If F is left exact in above sense, then it preserves monomorphisms and thus ~~X~~ $0 \to FX' \to FX \xrightarrow{\xi} FX''$ is exact if $0 \to X' \to X \to X''$ is exact. The converse also holds, clearly.

Similarly for right exactness.

c) A generalized definition drops the exactness condition on F and requires $F \longrightarrow L_0F$ with the universal property

$$\begin{tikzcd} & L_0F \\ F \arrow[ur] \arrow[dr] & \\ & \Phi_0 \arrow[uu, dashed, "\exists"'] \end{tikzcd} \Longrightarrow \begin{tikzcd} L_*F \\ \Phi_* \arrow[u, dashed, "\exists"'] \arrow[u, phantom, "\cong" near start, shift right=3] \end{tikzcd}$$

<u>Example D.1</u>. If F is exact, then the left and right derived functors of F are given by

$$L_0F = R^0F = F$$
$$L_iF = R^iF = 0 \qquad (i>0)$$

The only non-trivial thing to check is that the right square in

$$\begin{tikzcd} \Phi^0 X \arrow[r] & \Phi^0 X'' \arrow[r, "d_\Phi"] & \Phi^1 X' \\ FX \arrow[u] \arrow[r] & FX'' \arrow[u] \arrow[r, "d_F = 0"] & R^1FX' = 0 \arrow[u, "0"'] \end{tikzcd}$$

commutes and that the left square in

$$\begin{tikzcd} \Phi_1 X'' \arrow[r, "d_\Phi"] \arrow[d] & \Phi_0 X' \arrow[r] \arrow[d] & \Phi_0 X \arrow[d] \\ 0 = (L_1F)X'' \arrow[r] & FX' \arrow[r] & FX \end{tikzcd}$$

commutes, which can easily be checked using exactness of F.

<u>Definition D.2</u> An object I in an abelian category is <u>injective</u> iff the following equivalent conditions hold:

a) When $X \xrightarrow{\xi} Y$ is mono, any morphism $\iota: X \to I$ extends to some morphism $Y \longrightarrow I$

b) Any s.e.s. $0 \longrightarrow I \longrightarrow X \longrightarrow X'' \longrightarrow 0$ splits.

<u>Proof</u>: a) $\Rightarrow$ b) Extend Id_I to $\pi: X \longrightarrow I$.

b) $\Rightarrow$ a) Consider

$$0 \to I \longrightarrow \mathrm{Coker}(X \xrightarrow{(\iota,\xi)} I \oplus Y) \longrightarrow \text{cokernel} \longrightarrow 0$$

<u>Theorem D.1</u>. Let A be an Abelian category with sufficiently many injective objects.

a) Any left exact functor $A \longrightarrow B$ from A to an Abelian category B has a right derived functor.

b) Let $\Phi^*: \mathcal{A} \to \mathcal{B}$ be a cohomological functor. Then Φ^* is a right derived functor of Φ^0 iff $\Phi^i I = 0$ for all $i > 0$ and all injective objects I.

c) Let F. $0 \to \overline{\Phi}' \to \overline{\Phi} \to \overline{\Phi}'' \to 0$ be a sequence of left exact functors $\mathcal{A} \to \mathcal{B}$ and functor morphisms s.t.

$$0 \to \overline{\Phi}' I \to \overline{\Phi} I \to \overline{\Phi}'' I \to 0$$

is exact whenever I is an injective object of $\mathcal{A}$. Then there is a unique sequence of natural transformations $R^i \overline{\Phi}'' \to R^{i+1} \overline{\Phi}$ s.t.

$$0 \to \overline{\Phi}' X \to \overline{\Phi} X \to \overline{\Phi}'' X \to R^1 \overline{\Phi}' X \to$$

$$\cdots \to R^{i-1} \overline{\Phi}'' \to R^i \overline{\Phi}' X \to R^i \overline{\Phi} X \to R^i \overline{\Phi}'' \to R^{i+1} \overline{\Phi}' X \to \cdots$$

is exact for arbitrary $X \in \operatorname{Ob} \mathcal{A}$ and s.t.

$$\begin{array}{ccc}
R^i \overline{\Phi}'' X & \xrightarrow{d_F} & R^{i+1} \overline{\Phi}' X'' \\
\Big\downarrow {\scriptstyle d_{R^i \overline{\Phi}''}} & & \Big\downarrow {\scriptstyle d_{R \overline{\Phi}'}} \\
R^{i+1} \overline{\Phi}'' X' & \xrightarrow{d_F} & R^{i+2} \overline{\Phi}' X'
\end{array}$$

is anti-commutative.

<u>Proof</u>: "if"-part of b) Let $\Phi^p I = 0$ for all injective I and $p > 0$. Let $\alpha^0 : \overline{\Phi}^0 \to \Psi^0$ be given. By induction on ℓ we construct $\alpha^k : \overline{\Phi}^k \to \Psi^k$ for $k \leq \ell$ s.t.

$$\alpha^k_{F''} \, \delta_{F, \overline{\Phi}} = \delta_{F, \Psi} \, \alpha^{k-1}_{F''}$$

for $1 \leq k \leq \ell$, and a s.e.s. F: $0 \to F' \to F \to F'' \to 0$.

For $\ell = 0$ this is trivial.

Let $\ell > 0$ and assume α^k has been constructed for $k < \ell$. To construct α^ℓ, we consider any object X of $\mathcal{A}$ and choose a monomorphism $X \xrightarrow{\iota} I$ where I is injective.

When $\ell > 1$, we have as a part of the l.e.s for $0 \to X \to I \to \overset{\operatorname{coker}(\iota)}{\overset{\shortparallel}{X'}} \to 0$

$$\underset{\underset{0 \text{ when } \ell > 1}{\shortparallel}}{\overline{\Phi}^{\ell-1} I} \to \overline{\Phi}^{\ell-1} X' \xrightarrow{\delta_{\overline{\Phi}}} \overline{\Phi}^{\ell} X \to \underset{\underset{0}{\shortparallel}}{\overline{\Phi}^{\ell}} \text{ when } \ell \geq 1$$

giving us an isomorphism $\delta_{X,\iota} \, \overline{\Phi}^{\ell-1} X' \xrightarrow{\cong} \overline{\Phi}^\ell X$. We have $\delta_\Psi : \Psi^{\ell-1} X' \to \Psi^\ell X$. So we can put $\alpha^\ell_{X,\iota} = \delta_\Psi \, \alpha^{\ell-1}_{X'} (\delta_{X,\iota})^{-1}$

$$\begin{tikzcd} \Phi^{\ell-1}X' \arrow[r, "\delta_{X\iota}", "\cong"'] \arrow[d, "\alpha_X^{\ell-1}"'] & \Phi^{\ell}X \arrow[d, dashed, "\alpha^{\ell}_{X\iota}"] \\ \Psi^{\ell-1}X' \arrow[r] & \Psi^{\ell}X \end{tikzcd}$$

For $\ell = 1$ we still have

$$\mathrm{Coker}(\Phi^0 I \to \Phi^0 X') \xrightarrow[\cong]{\tilde{\delta}_{X\iota}} \Phi^1 X.$$

Also, α^0 defines

$$\mathrm{Coker}(\Phi^0 I \to \Phi^0 X') \xrightarrow{\tilde{\alpha}^0} \mathrm{Coker}(\Psi^0 I \to \Psi^0 X')$$

and we put $\alpha'_{X\iota} = \tilde{\delta}_\Psi \tilde{\alpha}^0 \tilde{\delta}_{X\iota}^{-1}$

$$\begin{tikzcd} \mathrm{Coker}(\Phi^0 I \to \Phi^0 X') \arrow[r, "\tilde{\delta}_{X\iota}", "\cong"'] \arrow[d, "\tilde{\alpha}^0"'] & \Phi^1 X \arrow[d, dashed, "\alpha'_{X\iota}"] \\ \mathrm{Coker}(\Psi^0 I \to \Psi^0 X') \arrow[r, "\tilde{\delta}_\Psi"] & \Psi^1 X \end{tikzcd}$$

where $\tilde{\delta}_\Psi$ is obtained from $\delta_\Psi : \Psi^0 X' \to \Psi^1 X$ by the universal property of $\mathrm{coker}(\Psi^0 I \to \Psi^0 X')$.

We now want to show that $\alpha^\ell_{X\iota}$ does not depend on ι and that $\alpha_X^\ell := \alpha^\ell_{X\iota}$ is a natural transformation $\Phi^\ell \to \Psi^\ell$. We can show both at the same time by considering monomorphisms $X \xrightarrow{\iota} I$, $Y \xrightarrow{k} K$, with injective objects I and K, and any morphism $X \xrightarrow{\xi} Y$ and showing

$$\Psi^\ell(\xi)\,\alpha^\ell_{X\iota} = \alpha^\ell_{Yk}\,\Phi^\ell(\xi)$$

When $Y = X$ and $\xi = id_X$, we have $\Phi^\ell(\xi) = id_{\Phi(X)}$ and $\Psi^\ell(\xi) = Id_{\Psi(X)}$, so $\alpha^\ell_{X\iota} = \alpha^\ell_{Xk}$, showing independence of ι for $\delta_{X\iota}$. The general case then ~~follows~~ $\int$ implies that α^ℓ is natural.

By the injectivity of K, there exists some morphism $\hat{\xi} : I \to K$ s.t. $\hat{\xi}\iota = k\xi$. We have an induced morphisms on cokernels giving a commutative diagram with ses as lows:

$$\begin{tikzcd} \ast : & 0 \arrow[r] & X \arrow[r, "\iota"] \arrow[d, "\xi"'] & I \arrow[r] \arrow[d, "\hat{\xi}"'] & X' \arrow[r] \arrow[d, "\xi'"'] & 0 \\ \mathcal{Y} & 0 \arrow[r] & Y \arrow[r] & K \arrow[r] & Y' \arrow[r] & 0 \end{tikzcd}$$

For $\ell > 1$ we have $\Phi^\ell(\xi)\,\delta_{X\iota} = \delta_{\Phi Y}\,\Phi^{\ell-1}(\xi')$, hence

$$\Psi^\ell(\xi)\,\alpha^\ell_{X,i} = \Psi^\ell(\xi)\,\delta_{\Psi,\mathcal{X}}\,\alpha^{\ell-1}_{X'}\,\delta_{X,i}^{-1}$$
$$= \delta_{\Psi,\mathcal{Y}}\,\Psi^{\ell-1}(\xi')\,\alpha^{\ell-1}_{X'}\,\delta_{X,i}^{-1}$$
$$= \delta_{\Psi,\mathcal{Y}}\,\alpha^{\ell-1}_{Y'}\,\Phi^{\ell-1}(\xi')\,\delta_{X,i}^{-1} \qquad \downarrow \text{(Induction)}$$
$$= \delta_{\Psi,\mathcal{Y}}\,\alpha^{\ell-1}_{Y'}\,\delta_{\Phi,\mathcal{X}}^{-1}\,\Phi^\ell(\xi) \qquad \downarrow (\Phi^\ell \text{ cohom functor})$$
$$= \alpha^\ell_{Y,k}\,\Phi^\ell(\xi),$$

proving the desired identity.
So α^ℓ is a well-defined natural transformation.
(For $\ell = 1$, we actually need a slight modification.)

Let $\mathcal{F} : 0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$ be a ses. We have to show that $\alpha^\ell_{X'}\,\delta_{\Phi,\mathcal{F}} = \delta_{\Psi,\mathcal{F}}\,\alpha^{\ell-1}_{X''}$.
Choose an ~~et~~ embedding $X' \longrightarrow I$ into some injective object, giving a commutative diagram

$$\begin{tikzcd}
\mathcal{F} & 0 \arrow[r] & X' \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, "i"] & X'' \arrow[r] \arrow[d, "i''"] & 0 \\
\tilde{\mathcal{F}} & 0 \arrow[r] & X' \arrow[r] & I \arrow[r] & \hat{X} \arrow[r] & 0
\end{tikzcd}$$

where i exists by injectivity of I and i'' is the induced morphisms on cokernels.
We have

$$\alpha^\ell_{X'}\,\delta_{\Phi,\mathcal{F}} = \alpha^\ell_{X'}\,\Phi^\ell(\mathrm{Id}_{X'})\,\delta_{\Phi,\mathcal{F}}$$
$$= \alpha^\ell_{X'}\,\delta_{\Phi,\mathcal{F}'}\,\Phi^\ell(i'')$$
$$= \delta_{\Psi,\mathcal{F}'}\,\alpha^{\ell-1}_{\hat{X}}\,\Phi^\ell(i'') \qquad \downarrow \text{construction of } \alpha^\ell$$
$$= \delta_{\Psi,\mathcal{F}'}\,\Psi^\ell(i'')\,\alpha^{\ell-1}_{X''} \qquad \downarrow (\alpha^{\ell-1} \text{ being a NT})$$
$$= \Psi^\ell(\mathrm{Id}_{X'})\,\delta_{\Psi,\mathcal{F}}\,\alpha^{\ell-1}_{X''}$$
$$= \delta_{\Psi,\mathcal{F}'}\,\alpha^{\ell-1}_{X''}$$

This completes the inductive construction of the α^ℓ and showing the existence part of the universal property of a right derived functor for Φ^*.
For uniqueness, let $(\alpha^\ell)_{\ell=0}^{\infty}$, $\alpha^\ell : \Phi^\ell \longrightarrow \Psi^\ell$ be any morphism of cohomological functors. Let X be any object and let $0 \longrightarrow X \xrightarrow{i} I \longrightarrow X' \longrightarrow 0$ be as before (so I injective).
As α^* is a morphism of cohomological functors it is compatible with

δ_Φ and δ_Ψ for this short exact sequence

$$\alpha^\ell_X \, \delta_\Phi = \delta_\Psi \, \alpha^{\ell-1}_{X'}$$

so

$$\alpha^\ell_X \, \delta_{X,2} = \delta_\Psi \, \alpha^{\ell-1}_{X'}$$

This shows that $\alpha^\ell_X = \delta_\Psi \alpha^{\ell-1}_{X'} \delta^{-1}_{X,2}$ (slight modification for $\ell = 1$) showing that α^ℓ is obtained from α^0 as above. So also the uniqueness part of the universal property follows.

The construction of derived functors uses injective resolutions and will yield the remaining part of Theorem D.1

<u>Definition D.3</u> An <u>injective resolution</u> of an object X of $\mathcal{A}$ is a long exact sequence

$$0 \longrightarrow X \xrightarrow{\ \xi\ } I^0 \xrightarrow{\ d^0\ } I^1 \xrightarrow{\ d^1\ } I^2 \longrightarrow \cdots$$

where all I are injective.

<u>Fact</u> In an Abelian category with sufficiently many injective objects, injective resolutions exist.

<u>Proposition D.1</u>. a) Let $\xi : X \longrightarrow I^*$ and $Y \xrightarrow{\ \nu\ } J^*$ be injective resolutions of the objects X and Y. If $f : X \to Y$ is any morphism, there is a morphism $\varphi : I^* \longrightarrow J^*$ compatible with f in the sense that $\nu f = \varphi^0 \xi$. If $\tilde\varphi$ is a different morphism of cochain complexes with the same property, there is a cochain homotopy $(s^\ell)_{\ell=0}^\infty$ $\quad s^\ell : I^\ell \longrightarrow J^{\ell-1}$ between φ and $\tilde\varphi$, i.e.

$$d_{J^*} s^\ell + s^{\ell-1} d_{I^*} = \varphi^\ell - \tilde\varphi^\ell$$

(where $s^0 = 0$.)

<u>Remark D.2</u> a) It follows that $\varphi(z)$ and $\tilde\varphi(z)$ differ by the boundary $d_{J^*} s^\ell(z)$ when $z \in Z^\ell(I^*)$, showing that φ and $\tilde\varphi$ induce the same morphism on cohomology.

b) The assumption that $\mathcal{A}$ has sufficiently many injective objects is not required. It is also sufficient if

$$0 \longrightarrow Y \longrightarrow J^0 \longrightarrow J^1 \longrightarrow \cdots$$

is a cochain complex with injective J^ℓ and $0 \to X \to I^0 \to I^1 \to$ an exact sequence where the I^ℓ may fail to be injective.

c) For ~~Abe~~ arbitrary Abelian category

$$\begin{array}{ccccc} & & Z^\ell(I^*) & & \\ & \overset{s^\ell}{\swarrow} & \downarrow {\scriptstyle \varphi - \tilde\varphi} & \overset{0}{\searrow} & \\ J^{\ell-1} & \xrightarrow{d_{J^*}} & Z^\ell(J^*) & \longrightarrow & H^\ell(J^*) = \operatorname{coker}(d_{J^*}) \end{array}$$

Proposition D.1 b) Let $X \xrightarrow{\xi} I^*$ and $Y \xrightarrow{\nu} J^*$ be injective resolutions. If $0 \longrightarrow X \longrightarrow Z \longrightarrow Y \longrightarrow 0$ is any ses, there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^0 \arrow[r] \arrow[d] & K^0 \arrow[r] \arrow[d] & J^0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^1 \arrow[r] \arrow[d] & K^1 \arrow[r] \arrow[d] & J^1 \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

where the rows are exact and, except for the first one, are split, and the columns are injective resolution

End of proof of theorem D.1

a) Let $F: \mathcal{A} \longrightarrow \mathcal{B}$ be left exact. For every object X in $\mathcal{A}$, choose an injective resolution $X \longrightarrow I_X^*$ of X. For every map $X \xrightarrow{f} Y$, choose an extension $I_X^* \xrightarrow{\hat{f}} I_Y^*$ of f.

Define

$$(R^pF)(X) = H^p(F I_X^*)$$

$$(R^pF)(f) = H^p(F I_X^* \xrightarrow{\hat{f}} F I_Y^*)$$

It follows from proposition D.1 a) that $R^pF(id_X) = id_{R^pFX}$.

For $X \xrightarrow{f} Y \xrightarrow{g} Z$, the maps $I_X^* \underset{\widehat{gf}}{\overset{\hat{g}\hat{f}}{\rightrightarrows}} I_Z^*$ are chain homotopic by proposition D.1 a). Applying the additive functor F gives the cochain homotopic morphisms $F(\hat{g})F(\hat{f})$ and $F(\widehat{gf})$, showing $R^pF(g) \circ R^pF(f) = R^pF(gf)$. It follows that R^pF is a functor.

To obtain the long exact cohomology sequence, consider a ses $0 \longrightarrow X \xrightarrow{\alpha} Z \xrightarrow{\beta} Y \longrightarrow 0$ and construct a ses $0 \longrightarrow I_X^* \xrightarrow{\hat{\alpha}} K^* \xrightarrow{\hat{\beta}} I_Y^* \longrightarrow 0$ by applying proposition D.1 b). We obtain a short exact sequence

$$0 \longrightarrow F I_X^* \longrightarrow F K^* \longrightarrow F I_Y^* \longrightarrow 0,$$

and thus a long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & R^{p-1}FX \arrow[r, "\delta"] & R^pFX \arrow[r, "H^p(\hat{\alpha})"] \arrow[dr, dashed, "R^pF(\alpha)"'] & H^p(FK^*) \arrow[r, "H^p(\hat{\beta})"] \arrow[d, "\cong"] & R^pFY \arrow[r, "\delta"] & R^{p+1}FX \\
 & & & R^pFZ \arrow[ur, dashed, "R^pF(\beta)"'] & &
\end{tikzcd}$$

The vertical morphism is obtained by choosing morphisms $k_1 : K^* \longrightarrow I_2^*$ and $k_2 : I_2^* \longrightarrow K^*$ (applying proposition D.1 a) twice to id_Z).

Applying F gives

$$F k_1 : FK^* \longrightarrow FI_2^*$$

$$F k_2 : FI_2^* \longrightarrow FK^*$$

s.t. $F(k_1 k_2)$ and $F(k_2 k_1)$ are chain homotopic to $id_{FI_2^*}$ resp id_{FK^*}. It follows that Fk_1 and Fk_2 induce isomorphisms on cohomology which are inverse to each other, defining the vertical map in the diagram.

To verify the commutativity of the diagram, one notes that $\hat{\alpha}$ and $k_1 \hat{\alpha}$ (and hence $F\hat{\alpha}$ and $F(k_1\hat{\alpha})$) are cochain homotopic by prop D.1 a), as are $\hat{\beta}$ and $\hat{\beta} k_2$ (and hence $F\hat{\beta}$ and $F(\hat{\beta} k_2)$)

Thus the long exact cohomology sequence has been established.

In a similar fashion, one can show its functorality on the category of s.e.s. in $\mathcal{A}$.

To show that R^*F is a right derived functor of F, we apply the "if"-part of theorem D.1 b), which we already proved.

Let X be an injective object. Then we can choose $I^* = (X \longrightarrow 0 \longrightarrow \ldots)$ and $X \xrightarrow{id_X} I^*$ as injective resolution. By prop D.1 a), there are ~~the~~ cochain morphisms $I_X^* \longrightarrow I^* \longrightarrow I_X^*$ compatible id_X that up to cochain homotopy are inverse to each other, showing

$$R^pFX = H^p(FI_X^*) = H^p(FI^*) = 0 \quad \text{for } p>0$$

Also $(R^0F)(X) = \ker(FI^0 \longrightarrow FI^1) \cong \cancel{\ker} F(\ker(I^0 \longrightarrow I^1))$

$$\cong F(X)$$

So by theorem D.1 b), R^*F is a right derived functor of F.

Theorem D.1 b "only if"

Let Φ^* be a right derived functor of $F = \Phi^0$. Then $\Phi^* \cong R^*F$ by the universal property of derived functors.

By the above

$$\Phi^p X = R^pFX = 0 \qquad \text{when } X \text{ injective.}$$

c) Let $0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$ be a sequence of left exact functors $\mathcal{A} \longrightarrow \mathcal{B}$ which is [illegible] on injective objects.

Choosing I_X^* as in the previous proof, we get a ses

$$(*) \qquad 0 \longrightarrow F' I_X^* \longrightarrow F I_X^* \longrightarrow F'' I_X^* \longrightarrow 0$$

and gives a l.es. on cohomology which is functorial in X as $(*)$ is.

The anti-commutativity of the connecting morphisms (with $0 \longrightarrow X \longrightarrow Z \longrightarrow Y \longrightarrow 0$) comes from the analogous (and well-known) assertion for the connecting morphisms in the square

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & F' I_X^* \arrow[r] \arrow[d] & F' K^* \arrow[r] \arrow[d] & F' I_Y^* \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F I_X^* \arrow[r] \arrow[d] & F K^* \arrow[r] \arrow[d] & F I_Y^* \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F'' I_X^* \arrow[r] \arrow[d] & F'' K^* \arrow[r] \arrow[d] & F'' I_Y^* \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

□

<u>Proof of proposition D.1</u>

a) Let $B^k \subseteq I^k$ be the image of $I^{k-1} \xrightarrow{d} I^k$ (resp $X \xrightarrow{\xi} I^0$ for $k=0$). As $B^0 \cong X$, we have a morphism $B^0 \xrightarrow{\tilde{\varphi}^0} J^0$ s.t. $uf = \tilde{\varphi}^0 \xi$. Let $I^j \xrightarrow{\varphi^j} J^j$ $(j<k)$ and $B^k \xrightarrow{\tilde{\varphi}^k} J^k$ s.t.

$$d_J \varphi^{j-1} = \varphi^j d_I \qquad 0 \leq j < k$$

and

$$d_J \varphi^{k-1} = \tilde{\varphi}^k d_I$$

(with $\varphi^{-1} = f$, $(d_I : I^{-1} \to I^0) = (X \xrightarrow{\xi} I^0)$, $(J^{-1} \to J^0) = (Y \xrightarrow{\nu} J^0)$) already be constructed. Let $I^k \xrightarrow{\varphi^k} J^k$ be an extension of $B^k \xrightarrow{\tilde{\varphi}^k} J^k$, which exists as J^k is injective. We have $d_J \varphi^k d_I = d_J \tilde{\varphi}^k d_I = d_J d_J \varphi^{k-1} = 0$, hence $d_J \varphi^k : I^k \longrightarrow J^{k+1}$ factors over $\operatorname{coker}(I^{k-1} \xrightarrow{d_I} I^k)$ $\cong I^k / B^k = I^k / Z^k \cong B^{k+1}$ (as $0 \to X \to I^*$ is acyclic)

This gives us a morphism $B^{k+1} \xrightarrow{\tilde{\varphi}^{k+1}} J^{k+1}$ with the desired property $\tilde{\varphi}^{k+1} d_I = d_J \varphi^k$.

The "existence" part of proposition D.1 (a) follows.

The "uniqueness" part will only be sketched. Wlog $\hat{\varphi} = 0$ and we need s with $\varphi = ds + sd$.

W.l.o.g. $f = 0$, so φ^0 vanishes on B^0, giving

$$B' = C^0/B^0 \xrightarrow{\tilde{s}_1} J^0$$

of which any extension s_1 satisfies $0 + s'd = \varphi^0$.

Then extend this further.

For a sketch of the proof of b):

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "\alpha"] \arrow[d] & Z \arrow[r, "\beta"] \arrow[d, dashed, "\binom{x}{\psi\beta}"] & Y \arrow[r] \arrow[d, "\psi"] & 0 \\
0 \arrow[r] & I^0 \arrow[r] & I^0 \oplus J^0 \arrow[r] & J^0 \arrow[r] & 0
\end{tikzcd}$$

where x is any extension of $X \to I^0$ to $Z \to I^0$.

<u>Theorem D.2</u> Let A be an abelian category with sufficiently many projective objects.

a) Every right exact functor $F: A \to B$ has a left derived functor L_*F.

b) A homological functor $\Phi_*: A \to B$ is a left derived functor iff $\Phi_p P = 0$ for all $p>0$ and P projective.

c) For a short sequence

$$0 \to F' \to F \to F'' \to 0$$

which is exact on projectives, one has a long exact sequence

$$\cancel{L_{p+1}F''A} \to L_pF'A \to L_pFA \to L_pF''A \to$$

$$\to L_{p-1}F'A \to L_{p-1}FA \to L_{p-1}F''A \to$$

$$\cdots$$

$$\begin{array}{ccccccc}
\to L_0F'A & \to & L_0FA & \to & L_0F''A & \to & 0 \\
\wr\| & & \wr\| & & \wr\| & & \\
F'A & & FA & & F''A & &
\end{array}$$

Construction of Ext

If A has sufficiently many injective objects, then for a fixed object X one has a right derived functor $Ext^*(X,Y)$ of the functor

$$(*) \qquad Y \longmapsto Hom(X,Y)$$

As $(*)$ is functorial in X, the $Ext^*(X,-)$ are also functorial in X. For a s.e.s. $0 \to X' \to X \to X'' \to 0$ and Y injective, one has a s.e.s.

$$0 \to Hom(X'',Y) \to Hom(X,Y) \to Hom(X',Y) \to 0$$

One thus has a long exact cohomology ~~the~~ sequence

$$(\%) \quad \begin{array}{l} 0 \to Ext^0(X'',Y) \to Ext^0(X,Y) \to Ext^0(X',Y) \to \\ \to Ext^1(X'',Y) \to Ext^1(X,Y) \to Ext^1(X',Y) \to \cdots \end{array}$$

by theorem 1.c)

Also for a s.e.s. $0 \to Y' \to Y \to Y'' \to 0$, by the structure of a derived functor we get the l.e.s.

$$(@) \quad \begin{array}{l} 0 \to Ext^0(X,Y') \to Ext^0(X,Y) \to Ext^0(X,Y'') \to \\ \to Ext^1(X,Y') \to Ext^1(X,Y) \to Ext^1(X,Y'') \to \end{array}$$

If we combine the six sequences thus obtained in a ~~commutative~~ diagram, the squares ~~are all~~ formed by the connecting morphism is anti-commutative.

We can construct Ext explicitly. Let ~~[illegible]~~ X and Y be objects in A and I^* an injective resolution of Y. Then by ~~definition~~ the construction of theorem D.i)

$$Ext^p(X,Y) = H^p(Hom(P, I^*))$$

For P projective, $Hom(P,-)$ is exact, so we indeed have

$$Ext^p(P,Y) = H^p(Hom(P, I^*)) = Hom(P, H^p(I^*)) = 0$$

When A also has sufficienty many projective objects, then for fixed Y the functor $A^{op} \longrightarrow$ (abelian groups) annihilates the injective objects of A^{op} (the projectives of A) in positive degrees hence is a right derived functor of

$$X \longmapsto Hom(X,Y)$$

Using the right derived functors of $X \longmapsto Hom(X,Y)$ also gives the l.e.s.'s $(\%)$ and $(@)$ now $(\%)$ by derived functor structure, $(@)$ from 1.c

When $\mathcal{A}$ has also sufficiently many injective objects, then by the previous argument these Ext-groups are canonically isomorphic to the ones obtained using injective resolutions of Y.
In other words we can calculate $Ext(X,Y)$ in two ways.

▷ Let $Y \longrightarrow I^*$ be injective resolution

$$Ext^p(X,Y) = H^p(Hom(X,I^*))$$

▷ Let $P^* \longrightarrow X$ be a projective resolution

$$Ext^p(X,Y) = H^p(Hom(P_*,Y))$$

An alternative construction of Ext' is as follows let $Ext'(X,Y)$ be the "class of isomorphism classes" of short exact sequences

$$\mathcal{E}: 0 \longrightarrow Y \longrightarrow E \longrightarrow X \longrightarrow 0$$

To such a s.e.s. one may associate $c_\mathcal{E}$, the image of id_X under

$$\to Hom(X,E) \to Hom(X,X) \longrightarrow Ext'(X,Y) \to Ext'(X,E) \to$$

or ${}_\mathcal{E}c$, the image of id_Y under

$$\to Hom(E,Y) \to Hom(Y,Y) \longrightarrow Ext'(X,Y) \to Ext'(E,Y) \to$$

Proposition: When $\mathcal{A}$ has sufficiently many injective or projective objects then $c_\mathcal{E} = {}_\mathcal{E}c$, and one has a bijection between the isomorphism classes of extensions and $Ext'(X,Y)$.

~~Rem~~ Remark a) The construction may have set-theoretic difficulties.
e.g. if $\mathcal{A} = \{(X,S,f)\}$ where X is an Abelian group, S a set and $f: S \longrightarrow End(X)$ any function s.t. two elements in the image of f commute, and $Hom_\mathcal{A}((X,S,f),(Y,T,g))$ is the set of all homomorphisms $\xi: X \longrightarrow Y$ of abelian groups s.t. $\xi f(s) = g(s)\xi$ when $s \in S \cap T$, and $\xi f(s) = 0$ when $s \in S \setminus T$; $g(t)\xi = 0$ when $t \in T \setminus S$.

Then there is no set of extensions of $(\mathbb{Z}, \emptyset, \emptyset)$ by itself intersecting all isomorphism classes of such extensions.

b) The higher order version of the construction of Ext' using extensions uses longer exact sequences (Yoneda-ext)

c) An isomorphism of extensions of X by Y is a comm diagram

$$\begin{tikzcd}
\xi: & 0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, "\cong"] & X \arrow[r] \arrow[d, equal] & 0 \\
\xi': & 0 \arrow[r] & Y \arrow[r] & E' \arrow[r] & X \arrow[r] & 0
\end{tikzcd}$$

<u>Corollary</u> If $\mathrm{Ext}^1(X,Y) = 0$, then every s.e.s.

$$0 \longrightarrow Y \longrightarrow E \longrightarrow X \longrightarrow 0$$

splits.
(Also follows from le. cohom. s.)

<u>Proposition D.2</u>. Let $F: \mathcal{A} \longrightarrow \mathcal{B}$ be a left exact functor between Abelian categories where $\mathcal{A}$ has sufficiently many injective objects. Let $\mathcal{X}$ be a class of objects of $\mathcal{A}$ with the following properties:

a) If $X \cong Y \oplus Z$, then $X \in \mathcal{X}$ iff $Y \in \mathcal{X}$ and $Z \in \mathcal{X}$.

b) Every object of $\mathcal{A}$ has a monomorphism into some element of $\mathcal{X}$

c) If $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ is exact and $X, Y \in \mathcal{X}$, then $Z \in \mathcal{X}$ and $0 \longrightarrow FX \longrightarrow FY \longrightarrow FZ \longrightarrow 0$ is exact.

Then $\mathcal{X}$ contains all injective objects and $R^pFX = 0$ when $X \in \mathcal{X}$.

<u>Proof</u>: Let I be injective. There is a monomorphism $I \longrightarrow X$ with $X \in \mathcal{X}$. By the injectivity of I, id_I extends to a projection $\pi: X \longrightarrow I$

$$\begin{tikzcd} I \arrow[r, "id_I"] \arrow[d] & I \\ X \arrow[ur, "\pi"'] & \end{tikzcd}$$

yielding an isomorphism $X \cong I \oplus \ker(\pi)$ and $I \in \mathcal{X}$ by (a).

If $X \in \mathcal{X}$, and $0 \longrightarrow X \longrightarrow I^*$ is an injective resolution, let $B^p = \ker(I^p \longrightarrow I^{p+1})$ (thus $B^0 \cong X$ and $B^{p+1} \cong \mathrm{im}(I^p \longrightarrow I^{p+1})$ when $p \in \mathbb{N}$.

Applying c) to the s.e.s.

$$0 \longrightarrow B^p \longrightarrow I^p \longrightarrow B^{p+1} \longrightarrow 0$$

shows $B^p \in \mathcal{X}$ by induction on p and gives the s.e.s.

$$0 \longrightarrow FB^p \longrightarrow FI^p \longrightarrow FI^{p+1} \longrightarrow 0$$

which may be spliced together to show exactness of

$$0 \longrightarrow FX \longrightarrow FI^*.$$

This ends ~~sub~~-appendix D.

Proposition 1.1.3 : a) For any PID R, $\mathrm{Ext}^p_R(M,N) = 0$ for $p > 1$

b) Any submodule of a projective R-module is projective. (R PID)

Proof a) The quotient of any injective R-module being injective, when $0 \to N \to I$ is exact, then

$$0 \to N \to I \to I/N \to 0$$

is an injective resolution of length 1 of N.

b) Let P be projective, $Q \subseteq P$ any submodule. We have the exact sequence

$$\underset{\substack{\| \\ 0}}{\mathrm{Ext}^1_R(P,T)} \to \mathrm{Ext}^1_R(Q,T) \to \underset{\substack{\| \\ 0}}{\mathrm{Ext}^2_R(P/Q,T)}$$

The first term is zero as P is projective and the last term is zero by a).

So $\mathrm{Ext}^1_R(Q,T) = 0$ for arbitrary R-modules T.

Applying this to $T = X'$ in a s.e.s.

$$0 \to X' \to X \to X'' \to 0$$

shows the exactness of $\mathrm{Hom}_R(Q,-)$, hence the projectivity of Q

Remark : a) We will soon generalize this to Dedekind domains.

b) For $a \in R \setminus \{0\}$, $\mathrm{Id}_{R/aR}$ does not lift to $R/aR \to R$, showing non-projectivity of R/aR. By classification of f.g. R-modules and the fact that any direct summand of a project module is projective, this shows that a f.g. R-module is projective iff it's free.

This can be seen to hold without the assumption of finite generation. Therefore "projective" in proposition 1.1.3 b) may be replaced by "free".

Example. If $a \in R$ is not a zero-divisor, then

$$0 \to R \xrightarrow{a} R \to R/aR \to 0$$

is a projective resolution of R/aR, so for M any R-module

$$\mathrm{Ext}_R^k(R/aR, M) = H^k(\mathrm{Hom}(R \xrightarrow{a} R, M)) = H^k(M \xrightarrow{a} M)$$

$$= \begin{cases} \ker(M \xrightarrow{a} M) & k = 0 \\ M/aM & k = 1 \\ 0 & k > 1 \end{cases}$$

<u>Proposition 1.1.4</u>: If R is any ring, then an R-module X is injective iff for all ideals $I \subseteq R$, $\mathrm{Ext}^1_R(R/I, X) = 0$

<u>Remark</u>. There is no similar criterion for projectivity, and by a famous result of Shelah the Whitehead problem "Is any Abelian group G with $\mathrm{Ext}^1_{\mathbb{Z}}(G, \mathbb{Z}) = 0$ free" undecidable in ZFC.

<u>Fact 1.1.1</u>: For $a \in R$ and R-modules M, N, the following coincide:

a) The multiplication $\cdot a$ on $\mathrm{Ext}^k_R(M, N)$ (which, as a ~~k~~ derived functor of a functor with values in R-modules is a functor to R-modules.)

b) The result of applying $\mathrm{Ext}^k_R(-, N)$ to $M \xrightarrow{\cdot a} M$

c) The result of applying $\mathrm{Ext}^k_R(M, -)$ to ~~M~~ $N \xrightarrow{\cdot a} N$

<u>Proof</u>. Using the fact that

$$\begin{array}{ccccc} 0 & \longrightarrow & N & \longrightarrow & I^* \\ & & \downarrow{\cdot a} & & \downarrow{\cdot a} \\ 0 & \longrightarrow & N & \longrightarrow & I^* \end{array}$$

may be used to calculate $\mathrm{Ext}^k_R(-, N \xrightarrow{a} N)$ we have that c) coincides with a).

Similarly one shows equality of a) and c) using projective resolutions

<u>Fact 1.1.2</u>. If multiplication by a annihilates M or N, then it annihilates $\mathrm{Ext}^k_R(M, N)$.

<u>Fact 1.1.3</u>: If R is a Noetherian ring, then any f.g. R-module M has a projective resolution

$$0 \longleftarrow M \longleftarrow P^0 \longleftarrow P^1 \longleftarrow P^2 \longleftarrow \cdots$$

where the P^i are finitely generated free R-modules.

<u>Proof</u>: There is a surjection $R^n \longrightarrow M$ for some $n \in \mathbb{N}$. R being Noetherian the kernel of this surjection is also finitely generated, so we may repeat the procedure.

<u>Remark</u> If R is coherent (any f.g. ideal is finitely presented) and M is a finitely represented R-module, the same holds.

(Recall that M is finitely presented if it may be written as a cokernel $R^m \longrightarrow R^n$.)

Using ~~the~~ $\exists$ such projective resolution

$$0 \longleftarrow M \longleftarrow R^{n_0} \longleftarrow R^{n_1} \longleftarrow R^{n_2} \longleftarrow \cdots$$

to calculate $\mathrm{Ext}^*(M, N)$ as

$$H^*(N^{m_0} \longrightarrow N^{m_1} \longrightarrow N^{m_2} \longrightarrow \dots)$$

one obtains:

Fact 1.1.4: If M and N are finitely generated modules over a Noetherian ring R, then the R-modules $\mathrm{Ext}_R^k(M,N)$ are all finitely generated.

We will now study the relation between localization and the Ext-groups.
Notice that for multiplicative $S \subseteq R$, we have functors

$$\{R\text{-modules}\} \underset{\text{forgetful}}{\overset{\text{localization}}{\rightleftarrows}} \{R_S\text{-modules}\}$$

where localization preserves projectivity and the forgetful functor preserves injectivity.
In fact, if $A \underset{R}{\overset{L}{\rightleftarrows}} B$ is an adjoint pair of functors, then L preserves projective objects when R is exact and R preserves injective objects when L is exact.
This may be applied ~~to~~ to the above situation.

Fact 1.1.5. If M is a projective R module and $S \subseteq R$ is a multiplicative subset, then M_S is a projective R_S-module. If N is an injective R_S module, it is injective as an R-module.

Fact 1.1.6: For any ring R and multiplicative sets $S \subseteq R$, we have a canonical isomorphism

(2) $$\mathrm{Ext}^k_{R_S}(M_S, X) \xrightarrow{\cong} \mathrm{Ext}^k_R(M, X)$$

where M is an R-module and X is an R_S module.
It is uniquely determined by its compatibility with the long exact cohomology sequences and by the condition that in degree 0 it coincides with the bijection

$$\mathrm{Hom}_{R_S}(M_S, X) \cong \mathrm{Hom}_R(M, X)$$

(the universal property of M_S)

Proof: When M is fixed, the rhs of (2) is a cohomological functor on the category of R_S-modules. By the universal property of the derived functor $\mathrm{Ext}^k_{R_S}(M_S, -)$, the bijection $\mathrm{Hom}(M_S, X) \cong \mathrm{Hom}(M, X)$ uniquely extends to a morphism (2). This is an isomorphism by ~~f~~ fact 1.1.5: the r.h.s. of (2) annihilates injective R_S modules X in positive degrees.

By Theorem 1, about derived functors, the rhs of (2) is also a right derived functor on the category of R_S-modules with the "same" degree 0 term, so they are canonically isomorphic.

<u>Note</u> on the compatibility of $\mathrm{Hom}(-,-)$ and $(-)_S$.

When M is an R-module, we have

$$\mathrm{Hom}_R(M,N) \longrightarrow \mathrm{Hom}_{R_S}(M_S, N_S)$$

sending

$$(f: M \to N) \longmapsto (f_S: M_S \longrightarrow N_S)$$

This is a morphism of R-modules ~~to~~ where the rhs is an R_S-module. By the universal property of localization this induces a unique morphism

$$(*) \qquad \mathrm{Hom}_R(M,N)_S \longrightarrow \mathrm{Hom}_{R_S}(M_S, N_S)$$

This in general fails to be injective or surjective.

When M is finitely generated this is injective

(If $f_S: M_S \to N_S$ is the zero morphism, then for all $m \in M$ there is $s \in S$ s.t. $s.f(m) = 0 \in N$. By finitely generatedness of M we can find a common annihilator $s \in S$ so $s.f(m) = 0$ for all $m \in M$, so $s.f = 0 \in \mathrm{Hom}_R(M,N)$, so f is zero in $\mathrm{Hom}_R(M,N)_S$.)

When M is finitely presented (e.g. M is f.g. and R is Noetherian), this is an isomorphism.

For instance when $M = \mathrm{coker}(R^m \longrightarrow R^n)$, then

$$M_S \cong \mathrm{coker}(R_S^m \longrightarrow R_S^n)$$

and

$$\mathrm{Hom}_R(M,N) \cong \ker(N^n \longrightarrow N^m)$$

and

$$\begin{aligned}\mathrm{Hom}_R(M,N)_S &\cong \ker(N^n \longrightarrow N^m)_S \\ &\cong \ker(N_S^n \longrightarrow N_S^m) \\ &\cong \mathrm{Hom}_{R_S}(M_S, N_S),\end{aligned}$$

which can be seen to coincide with $(*)$.

<u>Proposition 1.1.5</u>: Let R be a Noetherian ring, M a finitely generated R-module. Then for any R-module X and any multiplicative subset S, the canonical morphism

$$(3)\quad \mathrm{Ext}^*_R(M, X)_S \longrightarrow \mathrm{Ext}^*_{R_S}(M_S, X_S)$$

(obtained from $\mathrm{Hom}_R(A, B)_S \longrightarrow \mathrm{Hom}_{R_S}(A_S, B_S)$ and the universal property of derived functors) is an isomorphism.

The conditions characterizing (3) (for arbitrary R and M) are that in degree 0 it gives the homomorphism

$$\mathrm{Hom}_R(M, X)_S \longrightarrow \mathrm{Hom}_{R_S}(M_S, X_S)$$

and that it is compatible with the long exact sequences of Ext-modules in X.

<u>Proof</u> Existence and uniqueness of a morphism (3) satisfying the above conditions follows from the universal property of derived functors (giving $\mathrm{Ext}^*_R(M, X) \longrightarrow \mathrm{Ext}^*_{R_S}(M_S, X_S)$) followed by the universal property of localization.

To show that it is an isomorphism, one chooses a projective resolution

$$0 \longleftarrow M \longleftarrow R^{n_0} \longleftarrow R^{n_1} \longleftarrow \cdots$$

as in fact 1.1.3 and uses it to calculate both Ext^*-groups:

$$\begin{aligned}
\mathrm{Ext}^*_R(M, X)_S &\cong H^*(X^{n_0} \longrightarrow X^{n_1} \longrightarrow \cdots)_S \\
&\cong H^*(X_S^{n_0} \longrightarrow X_S^{n_1} \longrightarrow \cdots) \\
&\cong \mathrm{Ext}^*_{R_S}(M_S, X_S).
\end{aligned}$$

<u>Excursion on projective and injective dimension</u>

When $\mathcal{A}$ is an Abelian category and X any object, define

$$\mathrm{idim}_{\mathcal{A}}(X) = \sup\{p \in \mathbb{N} \mid \mathrm{Ext}^p(T, X) \neq 0 \text{ for some object } T \text{ of } \mathcal{A}\}$$

$$\mathrm{pdim}_{\mathcal{A}}(X) = \sup\{p \in \mathbb{N} \mid \mathrm{Ext}^p(X, T) \neq 0 \text{ for some object } T \text{ of } \mathcal{A}\}$$

(Either use Yoneda-ext or assume $\mathcal{A}$ has sufficiently many injective/projective objects).

When $\mathcal{A}$ has sufficiently many injectives, and $X \neq 0$, then

$$\begin{aligned}
\mathrm{idim}_{\mathcal{A}}(X) &= \min\{\ell \mid \text{There is an injective resolution of length } \ell\} \\
&= \min\{\ell \mid \text{Any injective resolution } 0 \to X \to I^* \text{ has truncated injective resolution } 0 \to X \to \tau_{\leq \ell} I
\end{aligned}$$

where this is ∞ if no such ℓ exists

We have denoted the soft truncation $\tau_{\leq k} C^*$ defined as

$$(\tau_{\leq k} C^*)^\ell = \begin{cases} C^\ell & \ell < k \\ Z^k = \ker(C^k \to C^{k+1}) & \ell = k \\ 0 & \ell > k \end{cases}$$

as opposed to

$$(\hat{\tau}_{\leq k} C^*)^\ell = \begin{cases} C^\ell & \ell \leq k \\ B^k = \mathrm{Im}(d : C^k \to C^{k+1}) & \ell = k+1 \\ 0 & \ell > k+1 \end{cases}$$

Proof of equality Let $\mathrm{idim}^{(a)}$, $\mathrm{idim}^{(b)}$, $\mathrm{idim}^{(c)}$ be the three definitions of $\mathrm{idim}_A(X)$ given above. Clearly

$$\mathrm{idim}^{(a)} \leq \mathrm{idim}^{(b)} \leq \mathrm{idim}^{(c)}$$

Assume $\mathrm{Ext}^p(T, X) = 0$ when $p > \ell$ and let $0 \to X \to I^*$ be any injective resolution of X, and let $Z^k = \ker(I^k \xrightarrow{d^k} I^{k+1})$. Then $Z^0 = X$ and the ses $0 \to Z^k \to I^k \to Z^{k+1} \to 0$ gives

$$\mathrm{Ext}^p(T, I^k) \to \mathrm{Ext}^p(T, Z^{k+1}) \to \mathrm{Ext}^{p+1}(T, Z^k) \to \mathrm{Ext}^{p+1}(T, I^k)$$

Since $\mathrm{Ext}^p(T, I^k) = 0$ for all k we see

$$\mathrm{Ext}^p(T, Z^{k+1}) \cong \mathrm{Ext}^{p+1}(T, Z^k),$$

so by induction we obtain $\mathrm{Ext}^p(T, Z^k) \cong \mathrm{Ext}^{p+k}(T, Z^0) \cong \mathrm{Ext}^{p+k}(T, X)$, for $p > 0$ and $k > 0$ or $p \geq 0$, $k = 0$. When $p > 0$ and $k = \ell$, this vanishes, showing injectivity of Z^ℓ and thus proving that $\tau_{\leq \ell} C^*$ is an injective resolution of X.

Dually, if A has sufficiently many projective objects, then

$$\begin{aligned} \mathrm{pdim}(X) &= \min\{\ell \mid \text{there is a proj. res. } 0 \leftarrow X \leftarrow P_* \text{ of length } \ell\} \\ &= \min\left\{\ell \;\middle|\; \begin{array}{l} \text{For any projective resolution } 0 \leftarrow X \leftarrow P_*, \\ 0 \leftarrow X \leftarrow P_0 \leftarrow \cdots \leftarrow P_{\ell-1} \leftarrow \ker(P_\ell \to P_{\ell-1}) \leftarrow 0 \\ \text{is a proj. res. of } X \end{array}\right\} \end{aligned}$$

By Baer's criterion, an R-module X is injective iff $\mathrm{Ext}^p(T, X) = 0$ when $p = 1$ and $T \cong R/I$ for some ideal I. By the above proof of equality of the different definitions of idim

$$\begin{aligned} \mathrm{idim}(X) &= \sup\{p \in \mathbb{N} \mid \mathrm{Ext}^p(R/I, X) \neq 0 \text{ for some ideal } I\} \quad (*) \\ &= \sup\{p \in \mathbb{N} \mid \mathrm{Ext}^p(T, X) \neq 0 \text{ for some f.g. } T\}, \end{aligned}$$

as it is sufficient to use such T as test objects for injectivity in our proof of $\mathrm{idim}^{(c)} \leq \mathrm{idim}^{(a)}$, hence the same proof also works when $\mathrm{idim}^{(a)}$ is replaced by $(*)$.

Using proposition 1.1.5:

Corollary 1.1.4 For any module X over a Noetherian ring R,

$$\operatorname{idim}_R(X) = \sup_{\mathfrak{m}\in m\text{-}\operatorname{Spec} R} \operatorname{idim}_{R_\mathfrak{m}}(X_\mathfrak{m}) = \sup_{\mathfrak{p}\in \operatorname{Spec} R} \operatorname{idim}_{R_\mathfrak{p}}(X_\mathfrak{p})$$

(We use: Any ideal of R_S is of the form $I\cdot R_S$ for some ideal I of R and $(R/I)_S \cong R_S/_{I\cdot R_S}$.)

(Also: if E is an R-module and $E_\mathfrak{m}=0$ for all $\mathfrak{m}\in m\text{-}\operatorname{Spec} R$ then $E=0$ since otherwise for $e\in E\setminus\{0\}$ $I=\operatorname{Ann}_R(e)$ is contained in some $\mathfrak{m}$

Corollary 1.1.5: For any finitely generated R-module M over a Noetherian ring R we have

$$\operatorname{pdim}_R(M) = \sup_{\mathfrak{m}\in m\text{-}\operatorname{Spec} R} \operatorname{pdim}_{R_\mathfrak{m}}(M_\mathfrak{m})$$

$$= \sup_{\mathfrak{p}\in \operatorname{Spec} R} \operatorname{pdim}_{R_\mathfrak{p}}(M_\mathfrak{p})$$

If R is a dedekind domain the $R_\mathfrak{m}$'s are fields or DVR's so are PID's and thus 1.1.4 and 1.1.5 may be applied

Corollary 1.1.6: A module M over a Dedekind domain R is injective iff it is divisible. Any quotient of an injective module is therefore injective and $\operatorname{idim}_R(M)\leq 1$ for all M.

Hence also $\operatorname{pdim}_R(M)\leq 1$ for all M and ~~thus~~ any submodule of a projective R-module is projective.

Any f.g. torsion-free R-module is projective.

Remark: The classification of f.g. modules ~~of~~ over Dedekind domains is

$$M \cong (R^{r-1}\oplus I)\oplus \bigoplus_{\mathfrak{m}\in m\text{-}\operatorname{Spec} R} \bigoplus_{k=1}^{\infty} (R/_{\mathfrak{m}^k R})^{e_{\mathfrak{m}k}}$$

where only finitely many $e_{\mathfrak{m}k}$ are non-zero and I is an ideal uniquely determined by M as is r.

There is a similar result about completion: If M is an R module and I is any maximal ideal, we define a topology (hence a uniform structure) on M with the $\mathfrak{m}^k M$ as a neighborhood base of 0.

We can complete it. (This involves passing to the largest Hausdorff quotient. It can be obtained as completion w.r.t. the semi-metric $d(m,n) = 2018^{-s}$ where $s=\sup\{k\in\mathbb{N} \mid m-n\in I^k M\}$.)

Call the completion $\hat{M}$. When R is Noetherian and M is f.g. and $N\subseteq M$ then $I^k M\cap N \supseteq I^{k-c}N$ for $k\geq c$, c suitably large by Artin-Rees. It follows that the I-adic topology on N is induced by the I-adic topology on M and the morphism $\hat{N} \longrightarrow \hat{M}$ between completions stays injective.

Since completion also commutes with forming quotients it is an exact functor on <u>finitely generated</u> R-modules.

When M is f.g. $\mathrm{Ext}^*(M,-)$ may be calculated using a resolution $0 \longleftarrow M \longleftarrow F_*$ by f.g. free modules. Then the $\hat{F}_i$ stay free (of the same rank) and moreover

$$\widehat{\mathrm{Hom}_R(F,N)} \cong \widehat{\mathrm{Hom}_R(R^\alpha, N)} \cong \widehat{N^\alpha} \cong (\hat{N})^\alpha$$
$$\cong \mathrm{Hom}_{\hat{R}}(\hat{F}, \hat{N})$$

Together with the exactness of $\wedge$ on f.g. R-modules, we obtain

<u>Proposition 1.1.6</u> If R is Noetherian and $I \subseteq R$ any ideal and M and N f.g. R-modules, then

$$\widehat{\mathrm{Ext}^p_R(M,N)} = \mathrm{Ext}^p_{\hat{R}}(\hat{M}, \hat{N}) \qquad (I\text{-adic completion})$$

1.2 Torsion products and flat modules

<u>Definition 1.2.1</u>: Let $\mathrm{Tor}_p^R(M,N)$ be the p-th left derived functor of $N \longmapsto M \otimes N$ from the category of R-modules to itself.

<u>Remark</u>: a) Recall that $M \otimes_R N$ has the universal property

$$\mathrm{Hom}_R(M \otimes_R N, T) \cong \mathrm{Bil}_R(M, N; T)$$

b) By definition, we have a long exact Tor-sequence for a given ses $0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$ of R-modules.

$$\longrightarrow \mathrm{Tor}^R_{p+1}(M, N'') \xrightarrow{\partial_{M,N}} \mathrm{Tor}^R_p(M, N') \longrightarrow \mathrm{Tor}^R_p(M,N)$$
$$\longrightarrow \mathrm{Tor}^R_p(M, N'') \xrightarrow{\partial_{M,N}} \mathrm{Tor}^R_p(M, N') \longrightarrow \dots$$

If M: $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is another ses of R-modules, then

$$0 \longrightarrow M' \otimes_R - \longrightarrow M \otimes_R - \longrightarrow M'' \otimes_R - \longrightarrow 0$$

is a ses of functors from R-modules to itself, which produces a sequence

$$(*) \qquad 0 \longrightarrow M' \otimes_R F \longrightarrow M \otimes_R F \longrightarrow M'' \otimes_R F \longrightarrow 0$$

when F is free.

(Indeed, when $F \cong R^\alpha$, $M \otimes_R F \cong M^\alpha$ <u>[coproduct</u> of α copies of M])

If P is a projective R-module, there is a free R-module F

and a surjection $F \xrightarrow{\pi} P$, which has an R-linear section $s: P \to F$ by projectivity of P. hence P is a direct summand of F, and $(*)$ also stays exact when F is replaced by P. Applying Theorem 1 c) about derived functors (actually its dual version) gives $\partial_{M,N} : \mathrm{Tor}_{p+1}^R(M'',N) \to \mathrm{Tor}_p^R(M',N)$, making the l.e.s. of Tor-modules.

Recall that we can calculate $\mathrm{Tor}_i^R(M,N)$ by choosing a projective resolution $P_*^{\ast} \to N$ and then

(1) $\quad \mathrm{Tor}_i^R(M,N) = H_i(M \otimes P_*)$

For short exact sequences $0 \to M' \to M \to M'' \to 0$ and $0 \to N' \to N \to N'' \to 0$ we have

(2) $\quad \mathrm{Tor}_{i+1}^R(M,N'') \longrightarrow \mathrm{Tor}_i^R(M,N')$

(3) $\quad \mathrm{Tor}_{i+1}^R(M'',N) \longrightarrow \mathrm{Tor}_i^R(M',N)$

where (2) is part of the definition of a homological functor and (3) is obtained by applying theorem 1c on derived functors to

$$0 \to M' \otimes - \longrightarrow M \otimes - \longrightarrow M'' \otimes - \longrightarrow 0$$

Fact 1.2.1: If $i > 0$ and one of M, N is projective, then $\mathrm{Tor}_i^R(M,N) = 0$

Proof: If N is projective, we can take $0 \to N \to N \to 0$ as a projective resolution, which is concentrated in degree 0. If $M = \coprod_S R$ (S any index set) is free, then $M \otimes P_* = \coprod_S P_*$ is acyclic in positive degrees, hence the vanishing.
For general projective M there is a free module F and a surjection $F \to M$ which splits by projectivity of M, making M a direct summand in F. So $\mathrm{Tor}_i^R(M,N)$ is a direct summand of $\mathrm{Tor}_i^R(F,N) = 0$ when $i > 0$. $\square$

It follows that for fixed N, $\mathrm{Tor}_i^R(-,N)$ has the properties characterizing derived functors (theorem 1 b)). It follows that $\mathrm{Tor}_i^R(-,N)$ are also the right derived functors of

$$M \longmapsto M \otimes N$$

Proposition 1.2.1: There is a unique family of isomorphisms $\mathrm{Tor}_i^R(M,N) \cong \mathrm{Tor}_i^R(N,M)$ which in $i=0$ coincide with $M \otimes N \cong N \otimes M$ and that exchanges (2) and (3). The composition $\mathrm{Tor}_i^R(M,N) \to \mathrm{Tor}_i^R(N,M) \to \mathrm{Tor}_i^R(M,N)$ is id.

Example 1.2.1. Let $a \in R$ be a non-zero divisor. Then
$$0 \longrightarrow R \xrightarrow{a} R \longrightarrow R/aR \longrightarrow 0$$
is a projective resolution of R/aR, so
$$\mathrm{Tor}_i^R(M, R/aR) \cong \begin{cases} M/aM & i=0 \\ \ker(M \xrightarrow{a} M) & i=1 \\ 0 & \text{otherwise} \end{cases}$$

In the same way as for Ext we have:

Fact 1.2.2: Let $r \in R$. The following endomorphisms of $\mathrm{Tor}_i^R(M,N)$ coincide.

a) Action of r on the R-module $\mathrm{Tor}_i^R(M,N)$

b) Applying $\mathrm{Tor}_i^R(M,-)$ to $N \xrightarrow{r\cdot} N$

c) Applying $\mathrm{Tor}_i^R(-,M)$ to $M \xrightarrow{r\cdot} M$.

When the ring R is Noetherian and $\mathrm{Tor}_i^R(M,N)$ and M are finitely generated. then the torsion products may be calculated using a free resolution $0 \leftarrow M \leftarrow R^{n_0} \leftarrow R^{n_1} \leftarrow \cdots$ by f.g. free R-modules: $\mathrm{Tor}_i^R(M,N) = H_i(N^{n_0} \leftarrow N^{n_1} \leftarrow \cdots)$.

Fact 1.2.3: If R is Noetherian and M, N are f.g. R-modules then for all i, $\mathrm{Tor}_i^R(M,N)$ is a finitely generated R-module.

Fact 1.2.4: If M, N are modules over a ring R and S is a multiplicative subset of R. there are unique isomorphisms
$$\mathrm{Tor}_i^R(M,N)_S \cong \mathrm{Tor}_i^R(M, N_S) \cong \mathrm{Tor}_i^R(M_S, N) \cong \mathrm{Tor}_i^R(M_S, N_S) \cong \mathrm{Tor}_i^{R_S}(M_S, N_S)$$
compatible with (2) and (3) and in degree $i=0$ equal to
$$(M \otimes_R N)_S \cong M \otimes_R N_S \cong M_S \otimes_R N \cong M_S \otimes_R N_S \cong M_S \otimes_{R_S} N_S.$$

Proof: The construction on the rhs are homological functors of M and N which annulate free R-modules and hence projective R-modules if $i>0$. $\mathrm{Tor}_i^R(M, N_S) = 0$ when M is projective. $\mathrm{Tor}_i^R(M_S, N) = 0$ when N is projective and $\mathrm{Tor}_i^{R_S}(M_S, N_S) = 0$ when M or N is projective. In degree 0 they are equal to the isomorphisms of tensor products mentioned above. By theorem 1(b) and the universal property of derived functors, they are canonically isomorphic:
$$\mathrm{Tor}_i^R(M,N)_S \cong \mathrm{Tor}_i^R(M_S, N) \cong \mathrm{Tor}_i^R(M, N_S) \cong \mathrm{Tor}_i^{R_S}(M_S, N_S).$$
The remaining isomorphism follows:
$$\mathrm{Tor}_i^R(M,N)_S \cong \mathrm{Tor}_i^R(M, N_S) \cong \mathrm{Tor}_i^R(M_S, N_S)$$

As $\otimes$ is compatible with direct limits $(M \otimes \varinjlim N_\lambda) \cong \varinjlim (M \otimes N_\lambda)$ and direct limits are exact ($H_*(\varinjlim C^\lambda_*) = \varinjlim H_*(C^\lambda_*)$)

[Franke remarks direct limits can be taken over more general index sets than only posets in which each two elements have a common "nominator":

a) For λ and ϑ there is η s.t. $\mathrm{Hom}_\Lambda(\lambda, \eta)$ and $\mathrm{Hom}_\Lambda(\vartheta, \eta)$ are both non-empty

b) For two morphisms $\lambda \rightrightarrows \vartheta$, there is a morphism $\vartheta \to \eta$ equalizing them]

it follows that:

<u>Fact 1.2.5</u>: Torsion products are compatible with direct limits

$$\mathrm{Tor}_i^R(M, \varinjlim N_\lambda) \cong \varinjlim \mathrm{Tor}_i^R(M, N_\lambda).$$

<u>Proposition 1.2.2</u>. For an R-module M, the following conditions are equivalent:

a) The functor $M \otimes -$ is exact

b) $\mathrm{Tor}_i^R(M, N) = 0$ when $i > 0$ and N any R-module $\hat{b})$ Only for $i = 1$

c) $\mathrm{Tor}_i^R(M, N) = 0$ when $i > 0$ and N f.g. $\hat{c})$ Only for $i = 1$

<u>Proof</u> a) $\Rightarrow$ b) By definition of $\mathrm{Tor}_i^R(M, N) \cong H_i(M \otimes P_*)$

b) $\Rightarrow$ c) trivial.

c) $\Rightarrow$ b) By fact 1.2.5

b) $\Rightarrow$ a) By the long exact sequence given by (2).

<u>Definition 1.2.2</u>. An R-module with these equivalent properties is called <u>flat</u>

<u>Proposition 1.2.3</u>. If R is a ring and M an R-module, t.f.a.e.

a) M is flat

b) For any ideal $I \subseteq R$, $\mathrm{Tor}_1^R(M, R/I) = 0$

c) For any ideal $I \subseteq R$, the morphism $I \otimes_R M \to IM$: $i \otimes m \mapsto im$ is an isomorphism

$\hat{c})$ For any ideal I, the morphism of c) is injective.

When R is Noetherian, each of $\hat{b})$ c) $\hat{c})$ may be weakened by requiring I to be a prime ideal.

<u>Proof</u>: a) $\Rightarrow$ b) trivial. c) $\Leftrightarrow$ $\hat{c})$ trivial, as $I \otimes_R M \to IM$ always epi.

b) $\Leftrightarrow$ $\hat{c})$ We have

$$\begin{tikzcd}
\mathrm{Tor}_1^R(R, M) \arrow[r] \arrow[d, phantom, "\shortparallel"] & \mathrm{Tor}_1^R(R/I, M) \arrow[r, "\partial"] & I \otimes M \arrow[r] \arrow[d, "\text{morphism from c)}"] & R \otimes M \arrow[d, "\cong"] \\
0 & & I \cdot M \arrow[r, hook] & M
\end{tikzcd}$$

so b) is equivalent to $\mathrm{im}(\partial) = 0$, which is equivalent to $\hat{c})$

b) $\Rightarrow$ a) We show proposition 1.2.2 c̃), i.e. we show by induction on the (finite) number n of generators of N that $\mathrm{Tor}_i^R(M,N)=0$.

When $n=0$ $N=0$ and this is trivial.

Let it be valid for fewer than n generators and let $N' \subseteq N$ be the submodule generated by the first $n-1$ generators. Then N/N' is generated by one element, hence isomorphic to R/I for some I. So then

$$\begin{array}{ccccc} \mathrm{Tor}_i^R(M,N') & \longrightarrow & \mathrm{Tor}_i^R(M,N) & \longrightarrow & \mathrm{Tor}_i^R(M,N/N') \\ \| & & & & \| \\ 0 & & & & 0 \end{array}$$

by induction and b) we have $\mathrm{Tor}_i^R(M,N)=0$.

Let R be Noetherian and b) satisfied for prime ideals. We will show $\mathrm{Tor}_i^R(M,N)=0$ for f.g. N, using the existence of a filtration

$$0 = N_0 \subseteq N_1 \subseteq \dots \subseteq N_n = N$$

with $N_i/N_{i-1} \cong R/\mathfrak{p}_i$ for some $\mathfrak{p}_i \in \mathrm{Spec}\, R$, which exists when R is Noetherian and N f.g. Then we can proceed as above, showing that $\mathrm{Tor}_i^R(M,N_i)=0$ by induction.

This is enough because equivalence of b) c) and c̃) holds for each individual I.

<u>Corollary 1.2.1</u>: A module M over a Dedekind domain R is flat if and only if it is ~~divisible~~ torsion-free.

<u>Proof</u>: When R is a PID, by proposition 1.2.3(b) all we need to check is that $\mathrm{Tor}_1^R(M, R/aR) = 0$ for all $a \in R$. By example 1.2.1, this is equivalent to M being torsion-free.

When R is an arbitrary Dedekind domain, the $R_\mathfrak{p}$ are all PID's for $\mathfrak{p} \in \mathrm{Spec}\, R$, so by the upcoming fact 1.2.6, M is flat iff all $M_\mathfrak{p}$ are flat iff $M_\mathfrak{p}$ torsion free $\forall \mathfrak{p}$ which is equivalent to M being torsion-free.

<u>Remark</u>: The proof shows that even if R is not Dedekind still M is flat $\Longrightarrow$ M is torsion free.

<u>Example 1.2.2</u>: Every projective module is flat.

<u>Proof</u> Fact 1.2.1 and proposition 1.2.3(b).

<u>Fact 1.2.6</u>: For an R-module the following are equivalent:

(a) M is flat

(b) For any <u>multiplicative set</u> $S \subseteq R$, M_S is a flat R-module

(c) ———— " ———— , M_S is a flat R_S-module

(d) For every maximal ideal $\mathfrak{m}$, $M_\mathfrak{m}$ is a flat R-module.
(e) ——— " ——— , $M_\mathfrak{m}$ is a flat $R_\mathfrak{m}$-module.

Proof. (a) ⟹ (b), (c) follows from proposition 1.2.3 and fact 1.2.4. Also (b) ⟹ (d) and (c) ⟹ (e) are trivial.
For (d), (e) ⟹ (a), if $I \subseteq R$ is any ideal and (d) or (e) holds, then
$$\mathrm{Tor}_1^R(M, R/I)_\mathfrak{m} \cong \mathrm{Tor}_1^R(M_\mathfrak{m}, R/I) \cong \mathrm{Tor}_1^{R_\mathfrak{m}}(M_\mathfrak{m}, R_\mathfrak{m}/I_\mathfrak{m}) = 0$$
for every maximal ideal of R by fact 1.2.4. So $\mathrm{Tor}_1^R(M, R/I) = 0$ for all I and thus M is flat, i.e. (a) holds.

Example 1.2.3. Over a Dedekind domain R with quotient field K, the modules K, $\prod_{i=1}^{\infty} R$ and any completion of R are flat (since they're torsion-free and we have corollary 1.2.1) but usually not projective.

Example 1.2.4: If $S \subseteq R$ is a multiplicative set, R_S is a flat R-module. This follows from fact 1.2.6 (b) since R is flat over itself. It also follows from the localization functor being exact.

Example 1.2.5: The completion $\hat{R}$ of a Noetherian ring R with respect to any ideal I is a flat R-module.
When M is a f.g. flat R-module, its completion $\hat{M}$ is a flat $\hat{R}$ module.

Proof. For f.g. R-modules M we have $M \otimes_R \hat{R} \cong \hat{M}$. Indeed this is true for $M = R$ and both sides commute with taking cokernels and finite direct sums since completion is exact on f.g. R-modules. Thus $- \otimes \hat{R}$ is an exact functor on f.g. R-modules. Applying this to any f.g. free resolution of R/I shows $\mathrm{Tor}_1^R(\hat{R}, R/I) = 0$ for any $I \subseteq R$, so $\hat{R}$ is flat.
Now for a f.g. flat R-module M, $\hat{M} \cong M \otimes_R \hat{R}$ is a tensor product of flat R-modules so is again a flat R-module. Since $- \otimes_{\hat{R}} \hat{M} = - \otimes_R M \otimes_R \hat{R}$, it is even a flat $\hat{R}$-module.

Example 1.2.6. The product of arbitrarily many flat modules over a Noetherian ring is again flat.

Proof: Let R be Noetherian and $(M_j)_{j \in J}$ a family of flat R-modules.
Let $I \subseteq R$ be any ideal and $0 \leftarrow R/I \leftarrow R^{n_0} \leftarrow R^{n_1} \leftarrow \cdots$ any free resolution by f.g. free modules. Then
$$\mathrm{Tor}_p^R\Big(\prod_{j \in J} M_j, R/I\Big) \cong H_p\Big(\prod_{j \in J} M_j^{n_0} \leftarrow \prod_{j \in J} M_j^{n_1} \leftarrow \cdots\Big)$$
$$\cong \prod_{j \in J} H_p\big(M_j^{n_0} \leftarrow M_j^{n_1} \leftarrow \cdots\big)$$
where the r.h.s vanishes by proposition 1.2.2 when $p > 0$. So by proposition 1.2.3 (b), $\prod_{j \in J} M_j$ is flat.

Example 1.2.7: The coproduct of arbitrarily flat modules is always flat, since $\bigoplus_{j \in J} -$ is exact and commutes with tensor products and hence with Tor.

<u>Fact 1.2.7</u>: For any R-module M, the following are equivalent

(a) $\mathrm{Tor}_p^R(M,T) = 0$ when $p > d$ for any R-module T

(a') $\mathrm{Tor}_{d+1}^R(M,T) = 0$ for any T

(b) $\mathrm{Tor}_p^R(M,T) = 0$ when $p > d$ for any T f.g.

(b') $\mathrm{Tor}_{d+1}^R(M,T) = 0$ for any f.g. T.

(c) $\mathrm{Tor}_p^R(M, R/I) = 0$ when $p > d$ for any ideal $I \subseteq R$

(c') $\mathrm{Tor}_{d+1}^R(M, R/I) = 0$ for any I.

Moreover, when R is Noetherian, it is enough to have (c) or (c') for prime ideals.

<u>Proof</u>: It is enough to show that (c') $\Longrightarrow$ (a). Suppose that (c') holds, resp. the weaker form of (c') when R is Noetherian. We can use the inductive argument from the proof of prop 1.2.3 to show that $\mathrm{Tor}_{d+1}^R(M,T) = 0$ for every f.g. R-module T and by fact 1.2.5 we can get rid of f.g.-ness, so we have (a').
Now assume that for some $p > d$, vanishing of $\mathrm{Tor}_p(M,-)$ has been shown. Let T be any R-module and choose an epimorphism $F \xrightarrow{\pi} T$ for some free R-module F. From the Tor-les associated to $0 \to \ker\pi \to F \to T \to 0$ we obtain

$$0 = \mathrm{Tor}_{p+1}^R(M,F) \to \mathrm{Tor}_{p+1}^R(M,T) \to \underset{\overset{\parallel}{0}}{\mathrm{Tor}_p^R(M,\ker\pi)}$$

hence $\mathrm{Tor}_{p+1}^R(M,T) = 0$. This shows (a).

<u>Definition 1.2.3</u>: Define the <u>flat dimension</u> $\mathrm{fl.dim}_R(M)$ of M to be the largest d such that the equivalent properties from fact 1.2.7 fail, or $+\infty$ if there is no such d, or $-\infty$ if $M=0$.

<u>Fact 1.2.8</u>: Let M be a module over an arbitrary ring R. Then

(a) $\mathrm{fl.dim}_R(M) \le \mathrm{pr.dim}_R(M)$

(b) $\mathrm{fl.dim}_R(M) = \sup\limits_{m \text{ max. id}} \mathrm{fl.dim}_{R_m}(M_m) = \sup\limits_{m} \mathrm{fl.dim}_R(M_m)$

(c) $\mathrm{fl.dim}_R(M) = \sup\limits_{\mathfrak{p} \text{ prime id}} \mathrm{fl.dim}_{R_\mathfrak{p}}(M_\mathfrak{p}) = \sup\limits_{\mathfrak{p}} \mathrm{fl.dim}_R(M_\mathfrak{p})$

<u>Proof</u>. When $\mathrm{pr.dim}_R(M) = \ell$, M has a projective resolution of length ℓ by fact 1.1.9. Using this resolution to compute $\mathrm{Tor}_\cdot^R(M,-)$, we see that $\mathrm{Tor}_p^R(M,-) = 0$ for $p > d$.
To show (b) and (c), use fact 1.2.4 and do the same as in the proof of corollary 1.1.5.

<u>Remark</u>: Examples 1.2.5, 1.2.6 and 1.2.7 remain true for the class of R-modules of flat dimension $\le d$.
In fact we can almost copy the proofs.

Proposition 1.2.4: For an R-module the following are equivalent:

(a) $\mathrm{fl.dim}_R(M) \leq d$

(b) M has a flat resolution of length d, i.e. an exact sequence

$$0 \leftarrow M \leftarrow F_0 \leftarrow \dots \leftarrow F_d \leftarrow 0 \qquad (4)$$

with flat F_i. When M is f.g. and R Noetherian, we may assume the F_i to be f.g. as well.

(c) For any sequence like (4) in which $F_0, \dots, F_{d-1}$ are flat, F_d is flat as well.

Proof. Put $B_i := \mathrm{im}(F_{i+1} \to F_i) = \ker(F_i \to F_{i-1})$ and $B_{-1} = M$. From (4) we get ses's $0 \to B_i \to F_i \to B_{i-1} \to 0$ for $0 \leq i \leq d-1$. Thus

$$\mathrm{Tor}^R_{p+1}(F_i, T) \to \mathrm{Tor}^R_{p+1}(B_{i-1}, T) \to \mathrm{Tor}^R_p(B_i, T) \to \mathrm{Tor}^R_p(F_i, T)$$

is exact for any R-module T. If $F_1, \dots, F_{d-1}$ are flat, the outer terms vanish, showing $\mathrm{Tor}^R_{p+1}(B_{i-1}, T) \cong \mathrm{Tor}^R_p(B_i, T)$. Since $F_d \cong B_{d-1}$, we thus get $\mathrm{Tor}^R_p(F_d, T) \cong \mathrm{Tor}^R_{p+d}(M, T)$. $\mathrm{Tor}_1^d(F_d, -) = 0$ iff $\mathrm{Tor}^R_{d+1}(M, -) = 0$.

Note that a resolution like (4) in which $F_0, \dots, F_{d-1}$ are flat always exists (e.g. soft-truncate a free resolution of M). In view of fact 1.2.7 (b) and proposition 1.2.2 (4), the equivalences follow.

Looking at the proof, we find

Fact 1.2.9: If $0 \to N \to F \to M \to 0$ is a ses of R-modules and F flat, then

$$\mathrm{fl.dim}_R(N) = \max\{\mathrm{fl.dim}_R(M) - 1, 0\}$$

And

Proposition 1.2.5. Let $\hat{(-)}$ denote the completion w.r.t. an arbitrary ideal I in a Noetherian ring R. For all f.g. R-modules M, N

$$\mathrm{Tor}_i^{\hat{R}}(\hat{M}, \hat{N}) \cong \widehat{\mathrm{Tor}_i^R(M, N)}$$

Proof. Choose a f.g. free resolution $0 \leftarrow N \leftarrow R^{n_0} \leftarrow R^{n_1} \leftarrow \dots$ of N. Then $0 \leftarrow \hat{N} \leftarrow \hat{R}^{n_0} \leftarrow \dots$ is a free resolution of $\hat{N}$ by exactness of $\hat{(-)}$ on f.g. R-modules. Thus

$$\begin{aligned}
\mathrm{Tor}_i^{\hat{R}}(\hat{M}, \hat{N}) &\cong H_i(\hat{M}^{n_0} \leftarrow \hat{M}^{n_1} \leftarrow \dots) \\
&\cong H_i(M^{n_0} \leftarrow M^{n_1} \leftarrow \dots)^\wedge \\
&\cong \mathrm{Tor}_i^R(M, N)^\wedge
\end{aligned}$$

since $\hat{(-)}$ commutes with f.g. homology by exactness on f.g. R-modules.

1.3 The case of f.g. modules over Noetherian rings

Proposition 1.3.1: Let P be a f.g. module over a Noetherian local ring R with maximal ideal m, $k = R/m$. Then t.f.a.e.

(a) P is free

(b) P is projective

(c) P is flat

(d) $\mathrm{Tor}_1^R(P, k) = 0$

(e) $\mathrm{Tor}_1^{\hat{R}}(\hat{P}, k) = 0$ where $\hat{(-)}$ denotes the completion w.r.t. m.

Proof: a) $\Rightarrow$ b) $\Rightarrow$ c) $\Rightarrow$ d) are clear.
For (d) $\Leftrightarrow$ (e), note that $\hat{R}$ is again local with residue field k. By corollary 1.1.7, ~~ever~~ an R-module vanishes iff its completion ~~of~~ ~~every~~ does, provided R is local. Then proposition 1.2.5 does the job.
For (d) $\Rightarrow$ (a), choose elements $p_1, \dots, p_n \in P$ whose images in $P/mP \cong P \otimes_R k$ form a basis of this k-vector space. By Nakayama's lemma, $p_1, \dots, p_n$ also generate P. Sending $e_i \mapsto p_i$ thus gives a surjection $R^n \xrightarrow{\pi} P$ which becomes an isomorphism when tensored with k. From (d) and the Tor les, we obtain

$$0 = \mathrm{Tor}_1^R(P, k) \to \ker(\pi) \otimes_R k \to k^n \to P \otimes_R k \to 0.$$

Since $k^n \to P \otimes_R k$ is an iso, this shows $\ker(\pi) \otimes_R k = 0$, hence $\ker(\pi) = 0$ by Nakayama's lemma. So π is already an isomorphism and R is free.

Corollary 1.3.1: For a f.g. module over a Noetherian local ring R we have

$$\mathrm{pr.dim}_R(M) = \mathrm{fl.dim}_R(M) = \sup\{d \in \mathbb{N}_0 \mid \mathrm{Tor}_{d+1}^R(M, k) \neq 0\}$$

Proof: The first equality is immediate from fact 1.1.9 and proposition 1.2.4(b) since flat and projective is the same for f.g. modules over Noetherian local rings by proposition 1.3.1.
The second equality follows from the proof of proposition 1.2.4, as in our situation it suffices to check $\mathrm{Tor}_1^R(F_d, k) = 0$ to show flatness of some f.g. R-module F_d by proposition 1.3.1(d).

Corollary 1.3.2: For any module M over a Noetherian local ring R,
$\mathrm{fl.dim}_R(M) \leq \mathrm{fl.dim}_R(k)$.

Proof. By ~~fact~~ proposition 1.2.1, $\mathrm{Tor}_{d+1}^R(k, M) \cong \mathrm{Tor}_{d+1}^R(M, k)$ and $\mathrm{fl.dim}_R(k)$ is the smallest d s.t. this vanishes, for all M. So by corollary 1.3.1, $\mathrm{fl.dim}_R(k) \geq \mathrm{fl.dim}_R(M)$.
If $p > \mathrm{fl.dim}_R(k)$ we thus have $p > \mathrm{fl.dim}_R(R/I)$ for all ideals $I \subseteq R$. So $\mathrm{Tor}_p^R(M, R/I) \cong \mathrm{Tor}_p^R(R/I, M) = 0$ and we get $\mathrm{fl.dim}_R(M) < p$ by fact 1.2.7(e)

Corollary 1.3.3. If M is a f.g. module over a Noetherian ring R (local or not), then $\mathrm{pr.dim}_R(M) = \mathrm{fl.dim}_R(M)$.

Proof. Follows from fact 1.2.8 and corollary 1.3.1

Proposition 1.3.2. For a f.g. Module over a Noetherian ring R. t.f.a.e.

(a) M is projective

(b) M is flat

(c) It is ~~cove~~ possible to cover $\mathrm{Spec}\, R$ by open subsets $\mathrm{Spec}\, R_f$ for $f \in R$ s.t. M_f is a free R_f-module.

(d) The sheaf of modules $\tilde{M}$ on $\mathrm{Spec}\, R$ is a vector bundle i.e. a locally free $\mathcal{O}_{\mathrm{Spec} R}$-module.

(e) $M_{\mathfrak{m}}$ is a free $R_{\mathfrak{m}}$-module for any maximal ideal $\mathfrak{m}$ of R

(f) $M_{\mathfrak{p}}$ —"— $R_{\mathfrak{p}}$-module —"— prime ideal $\mathfrak{p}$ of R.

Proof. By corollary 1.1.5. M is projective iff all $M_{\mathfrak{m}}$ or all $M_{\mathfrak{p}}$ are. The same holds for flatness by fact 1.2.8 (b) and (c) So by proposition 1.3.1. (a) (b). (e) and (f) are equivalent.

Equivalence of (c) and (d) is by definition.

(c) $\Rightarrow$ (f) is trivial and (f) $\Rightarrow$ (c) is a Nakayama-argument. using the following lemma

Lemma 1.3.1 Let M be a f.g. module over Noetherian ring R and $\mathfrak{p} \in \mathrm{Spec}\, R$. If $M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module. there is $f \in R \setminus \mathfrak{p}$ s.t. M_f is a free R_f-module.

Proof. Let $m_1, \dots, m_k \in M$ whose images in $M_{\mathfrak{p}}$ are free generators of $M_{\mathfrak{p}}/R_{\mathfrak{p}}$, and let $(g_i)_{i=1}^{k}$ be generators of M/R. Then in $R_{\mathfrak{p}}$. $g_i = \sum_{j=1}^{k} \rho_{ij} m_j$ with $\rho_{ij} \in R_{\mathfrak{p}}$. Since there are only finitely many ρ_{ij}, we have $\rho_{ij} = \frac{r_{ij}}{f}$ where $r_{ij} \in R$. $f \in R \setminus \mathfrak{p}$. ~~Red~~ Replacing R by R_f. M by M_f. ~~wina~~ wina $\rho_{ij} \in R$. Then for every $1 \leq i \leq K$ there is $f_i \in R \setminus \mathfrak{p}$ s.t. $f_i (g_i - \sum_{j=1}^{k} \rho_{ij} m_j) = 0$. Let $f = f_1 \cdots f_K$. and replace R by R_f. M by M_f. Then we may assume $g_i = \sum_{j=1}^{k} \rho_{ij} m_j$. Thus the m_{ij} generate M as an R-module.

Let $N \subseteq R^k$ be the kernel of $R^k \longrightarrow M : (r_i)_{i=1}^{k} \mapsto \sum_{i=1}^{k} r_i m_i$. We have a ses $0 \to N \to R^k \to M \to 0$ hence $0 \to N_p \to R_{\mathfrak{p}}^{k} \to M_{\mathfrak{p}} \to 0$, hence $N_{\mathfrak{p}} = 0$, as the m_i are free gen. of $M_{\mathfrak{p}}$. Let $(g_i)_{i=1}^{k}$ be generators of N, then each g_i becomes 0 in $N_{\mathfrak{p}}$ hence $f_i g_i = 0$ for some $f_i \in R \setminus \mathfrak{p}$. Then for $f = f_1 \cdots f_k$, $N_f = 0$.

2. Regular rings and Cohen-Macaulay-rings

2.1. An application of the Koszul complex

The Koszul complex $C_*(\underline{x}, M)$ is formed for an R-module M and a sequence $\underline{x} = (x_1, \dots, x_n) \in R^n$.

Examples (small n)

$$\begin{array}{lcccccc} & & 0 & & 1 & & 2 \\ C_*(\emptyset, M) & = & M & & & & \\ C_*((x_1), M) & = & M & \xleftarrow{\cdot x_1} & M & & \\ C_*((x_1,x_2), M) & = & M & \xleftarrow{(x_1, x_2)} & M^2 & \xleftarrow{\binom{x_2}{-x_1}} & M \end{array}$$

More formally let $I_n = \{1, \dots, n\}$. Define $C_p(\underline{x}, M)$ as

$$C_p(\underline{x}, M) = \{\text{Alternating functions } I^{n-p} \xrightarrow{f} M\}$$

where $f: I^{n-p} \longrightarrow M$ alternating means: (putting $k = n-p$)

1) $f(i_{\pi(1)}, \dots, i_{\pi(k)}) = \operatorname{sign}(\pi) f(i_1, \dots, i_k)$

2) $f(i_1, \dots, i_k) = 0$ if $(i_j)_{j=1}^k$ has repititions

The differential is given by

$$(df)(i_1, \dots, i_{k+1}) = \sum_{j=0}^{k} (d_j f)(i_1, \dots, i_{k+1})$$

where

$$(d_j f)(i_1, \dots, i_{k+1}) = x_{i_{j+1}} f(i_1, \dots, \widehat{i_{j+1}}, \dots, i_{k+1})$$

The study of $C_*(\underline{x}, M)$ (in the easiest case) is by using cones of morphisms of chain complexes.

Let $f: A_* \longrightarrow B_*$ be a morphism of chain complexes. Let

$$B_*[k]_i := B_{i+k} \quad \text{with} \quad d_{B[k]} = (-1)^k d_B$$

Let $C_i = \operatorname{Cone}(f)_i := B_i \oplus A_{i-1}$ with

$$d_{C_*}(b, a) = (d_B b + f(a), -d_A a)$$

This is a chain complex and there is a ses.

$$0 \longrightarrow B_* \longrightarrow \operatorname{Cone}(f)_* \longrightarrow A_*[-1] \longrightarrow 0,$$

hence

$$\begin{array}{ccccccc} \dots \longrightarrow H_i(B_*) \longrightarrow H_i(\operatorname{Cone}(f)) \longrightarrow & H_i(A_*[-1]) & \xrightarrow{\partial} & H_{i-1}(B_*) \\ & \| & & \| \\ & H_{i-1}(A_*) & \xrightarrow{f_*} & H_{i-1}(B_*) \end{array}$$

So we have a l.e.s.

$$\cdots \to H_i(A_*) \xrightarrow{f_*} H_i(B_*) \to H_i(\mathrm{cone}(f)) \to H_{i-1}(A_*) \to \cdots$$

Letting $\underline{x} = (x_1, \dots, x_n)$, $\underline{x}' = (x_1, \dots, x_{n-1})$, we have an iso

$$C_*(\underline{x}, M) \xrightarrow{\cong} \mathrm{Cone}\left(C_*(\underline{x}', M) \xrightarrow{\cdot x_n} C_*(\underline{x}', M) \right)$$

$$C_p(\underline{x}, M) \xrightarrow{\cong} C_p(\underline{x}', M) \oplus C_{p-1}(\underline{x}', M)$$

$$f \longmapsto (g, h)$$

$$g(i_1, \dots, i_k) = f(i_1, \dots, i_k), \quad h(i_1, \dots, i_{k-1}) = f(n, i_1, \dots, i_{k-1})$$

Thus the above l.e.s. turns into

$$0 \leftarrow H_0(\underline{x}, M) \leftarrow H_0(\underline{x}', M) \xleftarrow{x_n} H_0(\underline{x}', M) \leftarrow$$

$$H_1(\underline{x}, M) \leftarrow H_1(\underline{x}', M) \xleftarrow{x_n} H_1(\underline{x}', M) \leftarrow$$

<u>Definition 2.1.1</u>. A sequence $(x_i)_{i=1}^{n}$ is called <u>M-regular</u> if

$$M/(x_1M + \dots + x_{i-1}M) \xrightarrow{\cdot x_i} M/(x_1M + \dots + x_{i-1}M)$$

is injective for $1 \le i \le n$.

<u>Proposition 2.1.1</u>: a) If $\underline{x}$ is an M-regular sequence, then $H_i(\underline{x}, M) = 0$ when $i > 0$.

b) If R is Noetherian, M f.g. and $H_i(\underline{x}, M) = 0$ for $i > 0$. Then $\underline{x}$ is M-regular.

<u>Corollary 2.1.1</u>: When R is Noetherian local ring, M f.g., then any permutation of an M-regular sequence of elements of $\mathfrak{m}$ stays M-regular.

<u>Proof of proposition 2.1.1</u>. We prove in addition that for any $\underline{x}$:

α) $H_0(\underline{x}, M) = M/(x_1M + \dots + x_nM)$

For a) and α), we do induction on n. If $\underline{x} = \emptyset$, this is obvious. Let the assertion be valid for $\underline{x}'$. We have

$$0 \leftarrow H_0(\underline{x}, M) \leftarrow H_0(\underline{x}', M) \xleftarrow{x_n} H_0(\underline{x}', M)$$

showing α).

When $\underline{x}$ is M-regular, then $\underline{x}'$ is regular and we have

$$0 \leftarrow H_0(\underline{x}, M) \leftarrow H_0(\underline{x}', M) \xleftarrow{\cdot x_n} H_0(\underline{x}', M) \leftarrow H_1(\underline{x}, M)$$

(*)

$$H_1(\underline{x}, M) \leftarrow 0 \leftarrow 0 \leftarrow H_2(\underline{x}, M) \leftarrow 0 \leftarrow \dots$$

So $H_i(\underline{x}, M) = 0$ for $i \geq 2$ and for $i = 1$

$$H_1(\underline{x}, M) = \ker(H_0(\underline{x}', M) \xrightarrow{\cdot x_n} H_0(\underline{x}', M))$$

$$= \ker(M/\underline{x}'M \xrightarrow{\cdot x_n} M/\underline{x}'M)$$

$$M/(x_1M + \dots + x_{n-1}M)$$

$$= 0$$

by regularity of the sequence

We now prove b) in a similar fashion. For $\underline{x} = \emptyset$, this is trivial. Now let $n \geq 1$, and assume that $H_i(\underline{x}, M) = 0$ for $i > 0$. We first show that $\underline{x}'$ is M-regular. Otherwise $H_1(\underline{x}', M) \neq 0$ for some i by the induction hypothesis. By Nakayama's lemma and the fact that $x_n \in \mathfrak{m}$ then implies that $H_1(\underline{x}', M) \xrightarrow{\cdot x_n} H_1(\underline{x}', M)$ is not surjective. But by the above l.e.s. (*) we have

$$H_1(\underline{x}, M) \leftarrow H_1(\underline{x}', M) \xleftarrow{x_n} H_1(\underline{x}', M) \leftarrow H_2(\underline{x}, M)$$

so $H_{i+1}(\underline{x}, M) \neq 0$, a contradiction. So $\underline{x}'$ is M-regular.

Now the above arguments for a) may be reversed and (*) with $H_1(\underline{x}, M) = 0$ shows injectivity of $\cdot x_n$ on $M/\underline{x}'M$.

__Remark__: We have actually shown $H_1(\underline{x}, M) \neq 0$ when the $x_i \in \mathfrak{m}$, R Noetherian local, M f.g. and $\underline{x}$ not M-regular.

We obtain

__Fact 2.1.1__: Let R be a ring, $\underline{x} = (x_1, \dots, x_n)$ an R-regular sequence s.t. $I = \underline{x}R$ a proper ideal. Then

a) $0 \leftarrow R/I \leftarrow C_*(\underline{x}, R)$ is a free resolution of R/I.

b) For every R-module M

$$\mathrm{Tor}_i^R(M, R/I) \cong H_i(\underline{x}, M)$$

$$\mathrm{Ext}_R^i(R/I, M) = H_{n-i}(\underline{x}, M)$$

c) (Putting $M = R/I$) $\mathrm{prdim}(R/I) = \mathrm{fdim}(R/I) = n$.

2.2 Regular rings

Recall that if $(R, \mathfrak{m})$ is a Noetherian local ~~domain~~ (ring?) of dimension $d > 0$, then

$$\dim_k(\mathfrak{m}^i/\mathfrak{m}^{i+1}) \sim \varepsilon \cdot i^{d-1} \qquad (i \to \infty)$$

for some $\varepsilon > 0$, by Hilbert polynomial theory.

Here $k = R/\mathfrak{m}$. It follows that

$$\dim(\mathfrak{m}^2/\mathfrak{m}) \geq d$$

and when $\dim(\mathfrak{m}/\mathfrak{m}) = d$, there can be no relation in

$$Gr_\bullet(R, \mathfrak{m}) := \bigoplus_{i=0}^{\infty} \mathfrak{m}^i/\mathfrak{m}^{i+1}$$

between $\underline{x}_i = x_i \mod \mathfrak{m}^2$, where $(x_1 \dots x_\delta)$ is a base of $\mathfrak{m}/\mathfrak{m}^2$. If $\boxed{\dim(R) = \dim_k(\mathfrak{m}/\mathfrak{m}^2)}$, then R is called <u>regular</u>

It follows from the previous consideration that $Gr_\cdot(R, \mathfrak{m})$ is a polynomial ring, hence a domain. Therefore R is a domain.

Note that $Gr_\cdot(R/x_1R, \mathfrak{m}/x_1R) \cong Gr_\cdot(R, \mathfrak{m})/x_1 Gr_\cdot(R, \mathfrak{m})$

<u>Proposition 2.2.1</u>: Let R be a regular Noetherian local ring of dimension d and $(x_1, \dots, x_d)$ generators of $\mathfrak{m}$.

a) The sequence $\underline{x} = (x_1, \dots, x_d)$ is a regular sequence

b) The Koszul complex $C_*(\underline{x}, R)$ is a free resolution of length d for $k = R/\mathfrak{m}$

c) The <u>cohomological dimension</u> of the category of R-modules (i.e. supremum of projective or injective dimensions of R-modules) (also called the <u>global dimension</u> of R) is d.

<u>Proof</u>. The second assertion follows from the first one and the results of the previous subsection. The third follows from the second and corollary 1.3.2.

It remains to show a). Because regular local rings are domains, (x_1) is a regular sequence. Moreover, the images of $x_2, \dots, x_d$ in R/x_1R generate the maximal ideal. As $Gr_\cdot(R/x_1R, \mathfrak{m}/x_1R) \cong Gr_\cdot(R, \mathfrak{m})/x_1 Gr_\cdot(R, \mathfrak{m})$, and thus $Gr_\cdot(R/x_1R, \mathfrak{m}/x_1R)$ is a polynomial ring in free generators $x_2 \mod \mathfrak{m}^2, \dots, x_d \mod \mathfrak{m}^2$. Therefore $\dim Gr(R/x_1, \mathfrak{m}/x_1) \sim K^{d-2}$

So R/xR is regular of dimension $d-1$. The regularity of $\underline{x}$ now follows by induction on d.

<u>Theorem 1 (Serre)</u>

For a Noetherian local ring R, the following are equiv:

a) R is regular.

b) The ~~ch~~ cohomological dimension of the category of R-modules ~~equals dim(R)~~ is finite.

c) The cohomological dimension of the category of R-modules equals $\dim(R)$

d) $d := \mathrm{fdim}(k) < \infty$

<u>Proof</u> a) $\xRightarrow[\text{prop 2.2.1}]{}$ c) $\Longrightarrow$ b) $\Longrightarrow$ $\mathrm{pdim}(k) < \infty$ $\Longrightarrow$ d)

For d) $\Rightarrow$ a) we do induction on $d := \mathrm{fdim}(k)$.

Recall that

$$\mathrm{Ass}(M) = \{\mathfrak{p} \in \mathrm{Spec}\, R \mid \mathfrak{p} = \mathrm{Ann}(m) \text{ for some } m \in M\}$$

When R is Noetherian,

$$\{r \in R \mid M \xrightarrow{r} M \text{ is injective}\} = R \setminus \bigcup_{\mathfrak{p} \in \mathrm{Ass}(M)} \mathfrak{p}$$

Note that we have

- $\mathrm{Ass}(R/\mathfrak{p}R) = \{\mathfrak{p}\}$
- For a s.e.s. $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$

$$\mathrm{Ass}(M') \subseteq \mathrm{Ass}(M) \subseteq \mathrm{Ass}(M') \cup \mathrm{Ass}(M'')$$

- As any f.g. has filtration by finitely many $M_i \subseteq M$ s.t. $M_i/M_{i-1} \cong R/\mathfrak{p}_i$, ~~to~~ it follows that $\mathrm{Ass}(M)$ is finite.

Also recall <u>prime avoidence</u>.

> Let $I \subseteq R$ be an ideal not contained in the ideals $\mathfrak{p}_1 \dots \mathfrak{p}_d$ of which at most two fail to be prime
>
> Then $I \not\subseteq \bigcup_{i=1}^{d} \mathfrak{p}_i$

Back to the <u>proof</u>

If $d = 0$, $\mathrm{pdim}(k) = 0$ by corollary 1.3.1 and by prop 1.3.1 k is a free R-module. Hence $k = R$ is regular of dimension 0.

Let $d > 0$ and the assertion be shown for smaller d.

<u>Claim</u>: $\mathfrak{m} \notin \mathrm{Ass}(R)$.

Otherwise $R/\mathfrak{m} \cong k$ is a submodule of R. Thus there is some s.e.s.

$0 \to k \to R \to Q \to 0$. Thus

$$\mathrm{Tor}_{p+1}(T, Q) \to \mathrm{Tor}_p(T, k) \to 0$$

when T is any R-module and $p > 0$.

As $\mathrm{fdim}(Q) \leq \mathrm{fdim}(k)$ (corollary 1.3.2) and $\mathrm{fdim}(k) > 0$, it follows that $\mathrm{fdim}(k) < \mathrm{fdim}(k)$, a contradiction proving the claim.

Moreover, $\mathfrak{m} \neq \mathfrak{m}^2$ as otherwise $\mathfrak{m} = 0$ by Nakayama's lemma, so $R = k$ and $d = 0$.

Thus by prime avoidence, it is possible to choose

$$x \in \mathfrak{m} \setminus \Big(\mathfrak{m}^2 \cup \bigcup_{\mathfrak{p} \in \mathrm{Ass}\, R} \mathfrak{p}\Big)$$

Then $R \xrightarrow{x} R$ is injective. If $0 \leftarrow M \leftarrow F_*$ is a free resolution of some R-module M s.t. (x) is an M-regular sequence, then

$$0 \leftarrow M/xM \leftarrow F_*/xF_*$$

is a free resolution of the R/xR-module M/xM. Therefore

$$(*) \quad \mathrm{pdim}_{R/xR}(M/xM) \leq \mathrm{pdim}(M) \qquad (M \text{ f.g. over } R,\ M \xrightarrow{x} M \text{ inj.})$$

We apply this to $M = \mathfrak{m}_R$. By $0 \to \mathfrak{m} \to R \to k \to 0$, we have $\mathrm{Tor}_d^R(\mathfrak{m}, T) \cong \mathrm{Tor}_{d+1}(k, T)$ for any R-module T.

It follows that

$$\mathrm{pdim}_R(\mathfrak{m}) \overset{\text{cor 1.3.3}}{\leq} \mathrm{fdim}_R(\mathfrak{m}) < d.$$

By $(*)$,

$$(+) \quad \mathrm{pdim}_{R/xR}(\mathfrak{m}_R/x\mathfrak{m}_R) < d.$$

If $(x = x_1, \dots, x_e)$ is a sequence of elements of $\mathfrak{m} = \mathfrak{m}_R$, whose images in $\mathfrak{m}_R/\mathfrak{m}_R^2$ form a basis of that $k = (R/\mathfrak{m})$-vector space. Then we get a splitting of the ses

$$0 \to k \xrightarrow{\cdot x} \mathfrak{m}_R/x\mathfrak{m}_R \to \mathfrak{m}_R/xR \to 0$$

sending any element $m = \mu \bmod x\mathfrak{m}$ of $\mathfrak{m}/x\mathfrak{m}_R$ to c_1, the coefficient of $x = x_1$ in the expression

$$\mu \bmod \mathfrak{m}^2 = \sum_{i=1}^{e} c_i \cdot (x_i \bmod \mathfrak{m}^2)$$

in the above base of $\mathfrak{m}/\mathfrak{m}^2$. Thus k as an (R/xR)-module is a direct summand of $\mathfrak{m}/x\mathfrak{m}$.

Then by (+)

$$\operatorname{fdim}_{R/xR}(k) \leq \operatorname{pdim}_{R/xR}(k) \leq \operatorname{pdim}_{R/xR}(\mathfrak{m}/x\mathfrak{m}) < d.$$

By the induction assumption. R/xR is a regular local ring of dimension $\delta <\dim(R)$ (!) Therefore there are $\xi_1 \dots \xi_\delta \in \mathfrak{m}$ whose images in $\mathfrak{m}/xR$ generate that ideal of R/xR. So $(\xi_1 \dots \xi_\delta, x)$ are generators of $\mathfrak{m}$ as an ideal in R.

Thus $\dim(R) \leq \delta+1$ and combining this with $\delta < \dim(R)$ we get $\dim(R) = \delta+1$ and R is a regular local ring of dimension d by proposition 2.2.1.

It remains to prove (!) Let $\mathfrak{q}_1 \supsetneq \dots \supsetneq \mathfrak{q}_\delta$ be a sequence of prime ideals in R/xR with preimages $\mathfrak{p}_1 \supsetneq \dots \supsetneq \mathfrak{p}_\delta$ in R. Then $x \in \mathfrak{p}_i$. If $\mathfrak{p}_\delta$ contains another prime ideal of R, then $\dim(R) > \delta$ as claimed. This is indeed the case. If not, then $\mathfrak{p}_\delta$ is a minimal prime ideal of R But the minimal prime ideals of any Noetherian ring are always associated prime ideals. But x was chosen not to be contained in any associated prime ideal of R.

<u>Corollary 2.2.1 (Serre)</u>. Let A be a regular local ring and $\mathfrak{p} \in \operatorname{Spec} R$. Then $R_\mathfrak{p}$ is also a regular local ring.

<u>Proof</u>. As $k(\mathfrak{p})$ is the localization of $R/\mathfrak{p}$ w.r.t. $\mathfrak{p}$, its pdim as an $R_\mathfrak{p}$-module is bounded by the projective dimension of $R/\mathfrak{p}$ as an R-module, by corollary 1.1.5.

<u>Definition 2.2.1</u>: A Noetherian ring R is called <u>regular</u> if it satisfies the following equivalent condition.

a) For any $\mathfrak{p} \in \operatorname{Spec} R$, $R_\mathfrak{p}$ is a regular local ring.

b) For any $\mathfrak{m} \in \operatorname{mSpec} R$. $R_\mathfrak{m}$ is a regular local ring

Also R is called <u>regular in I</u>, where $I \subseteq R$ an ideal, if it satisfies the following equivalent conditions:

a) For any $\mathfrak{p} \in \operatorname{Spec} R$ containing I, $R_\mathfrak{p}$ is regular

b) For any $\mathfrak{m} \in \text{m-Spec}\, R$ containing I. $R_\mathfrak{m}$ is regular.

<u>Proposition 2.2.2</u>: Let R be a Noetherian ring regular in its ideal I. Then the completion $\hat{R}$ of R w.r.t. I is a regular ring.

Proof: For $x \in I \cdot \hat{R}$,

$$(1+x)^{-1} = \sum_{i=0}^{\infty} x^i$$

converges in $\hat{R}$, so $I \subseteq$ (Jacobson ideal of $\hat{R}$) $= \bigcap_{m \in \mathrm{mSpec}(\hat{R})} m$.

Hence any maximal ideal m of $\hat{R}$ contains I, so that its pre image n in R also is a maximal ideal and $m = \hat{n}$.

Then $Gr.(\hat{A}, m) \cong Gr.(A, n)$, implying that $\hat{R}_m$ and R_n have the same Krull dimension d, by Hilbert polynomial theory.

If n is generated by $x_1, \dots, x_d / R_n$, then so is $m = \hat{n}$ by their images, showing that $\hat{R}_m$ is regular.

Proposition 2.2.3. If R is a regular Noetherian ring, then $R[T_1, \dots, T_n]$ and $R[[T_1, \dots, T_n]]$ are also regular.

Proof: As $R[[T_1, \dots, T_n]]$ is the completion of $R[T_1, \dots, T_n]$ w.r.t. $\langle T_1, \dots, T_n \rangle$, it suffices to prove the first claim.

By induction, we may assume $n=1$. Thus we have to show that $R[T]$ is regular.

Fact: Let R be Noetherian.

a) All minimal prime ideals are in $\mathrm{Ass}(R)$

b) $\mathrm{Ass}(M_S) = \{ p \in \mathrm{Ass}(M) \mid p \cap S = \emptyset \}$

Proof: b) It is clear that "$\supseteq$" holds. If $q \in \mathrm{Ass}(M_S)$, let $q = \mathrm{Ann}(\mu)$, where $\mu = \frac{m}{s} \in M_S$. Without loss of generality, $\mathrm{Ann}_R(m)$ is maximal among all representatives of μ (using that R is Noetherian.) Then $\mathrm{Ann}_R(m)$ is prime: if $a, b \in R$ have $abm = 0$, then w.l.o.g. $am = 0$ in M_S as q is prime, so $sam = 0$ for some $s \in S$, but then $a \in \mathrm{Ann}_R(m)$ by choice of m. Thus $p = \mathrm{Ann}_R(m)$ is a ~~ge~~ prime ideal. Since $q = \mathrm{Ann}(\mu) = p R_S$, the assertion follows.

a) Let p be a minimal prime ideal, then

$\mathrm{Ass}(R_p) = \{ p R_p \}$, as this is the only prime ideal of R_p. By b), $p \in \mathrm{Ass}(R)$.

Proof of proposition 2.2.3

We continue proving that $R[T]$ is regular.

Let $p \in \mathrm{Spec}\, R[T]$, $m = p \cap R$. W.l.o.g. this is a maximal ideal (localize it otherwise, using Serre's theorem to keep regularity!)

and R is local. Let $x_1, \dots, x_d$ be generators of $\mathfrak{m}$. If $\mathfrak{p} = \mathfrak{m}R[T]$, then they also generate $\mathfrak{p}$, and $ht(\mathfrak{m}R[T]) \geq ht(\mathfrak{m})$, hence $ht(\mathfrak{m}R) = d$, and $\mathfrak{m}R[T]$ is generated by d elements.

Otherwise, $ht(\mathfrak{p}) > ht(\mathfrak{m})$ (as $\mathfrak{p} \supsetneq \mathfrak{m}[T]$ and the second is prime), and $R[T]/\mathfrak{p}$ is a ~~field~~ finite field extension of $k = k(\mathfrak{p})$. Let $Q \in R[T]$ s.t. Q mod $\mathfrak{m}$ is a minimal polynomial of T mod $\mathfrak{p}$, then the $Q, x_1, \dots, x_d$ generate $\mathfrak{p}$, hence $ht(\mathfrak{p}) \leq d+1$, and equality occurs.

Hence R is regular

<u>Proposition 2.2.4</u>: All f.g. modules over regular local rings have finite free resolutions

2.3 Regular sequences and depth

Let $(R, \mathfrak{m}, k)$ a Noetherian local ring.

<u>Proposition 2.3.1</u>. Let M be a f.g. R-module. For a natural number n, the following conditions are equivalent.

a) There is a f.g. R-module T s.t.

$$\mathrm{supp}(T) = \{\mathfrak{m}\} \qquad \mathrm{Ext}^j(T, M) = 0 \quad \text{when } j < n$$

b) $\mathrm{Ext}^j(k, M) = 0$ when $j < n$

c) For every f.g. R-module T with $\mathrm{supp}(T) = \{\mathfrak{m}\}$ we have $\mathrm{Ext}^j(T, M) = 0$ when $j < n$.

<u>Proof</u> c) $\Rightarrow$ b) $\Rightarrow$ a) are obvious

a) $\Rightarrow$ b) We do induction on n. case $n=0$ trivial. Let T be as in a). We have a filtration of T with quotients of the form $R/\mathfrak{p}$, and the $\mathfrak{p}$ must be $\mathfrak{m}$ as $\mathrm{Supp}(T) = \{\mathfrak{m}\}$. So there is a surjection $T \longrightarrow k$ and hence a s.e.s. $0 \longrightarrow T_1 \longrightarrow T \longrightarrow k \to 0$. We get a l.e.s.

$$\longrightarrow \mathrm{Ext}^j(T, M) \longrightarrow \mathrm{Ext}^j(k, M) \longrightarrow \mathrm{Ext}^{j-1}(T_1, M) \longrightarrow$$

The left term vanishes by assumption for $j < n$ and by the induction assumption $\mathrm{Ext}^{j+1}(T_{i-1}, M)$ vanishes for $j < n$ as well so $\mathrm{Ext}^j(k, M)$ vanishes.

b) $\Rightarrow$ c) If $0 \longrightarrow T' \longrightarrow T \longrightarrow T'' \longrightarrow 0$ and $\mathrm{Ext}^j(T', M) = \mathrm{Ext}^j(T'', M) = 0$, then $\mathrm{Ext}^j(T, M) = 0$ by the l.e.s. Using induction on the length of a filtration $0 = T_0 \subseteq T_1 \subseteq \dots \subseteq T_k = T$, with $T_i/T_{i-1} \cong R/\mathfrak{p}$ (hence $\mathfrak{p} = \mathfrak{m}$), we obtain $\mathrm{Ext}^j(T, M) = 0$ for $j < n$.

<u>Definition</u>. The largest such number n is called the <u>depth</u> (or <u>homological codimension</u>) of the module and is denoted depth(M), prof(M), codh(M), or ht(M).

<u>Fact 2.3.1</u> $\mathrm{Depth}(A \oplus B) = \min(\mathrm{depth}(A), \mathrm{depth}(B))$

<u>Remark 2.3.1</u> $\mathrm{Supp}(M) = \{\mathfrak{m}\}$ is equivalent to $\dim(T) = 0$. Under this condition we have $\mathrm{Ext}^j(T, M) = 0$ when $j < \mathrm{depth}(M)$.

<u>Theorem 2</u> (Auslander - Buchsbaum) Let R be a Noetherian local ring and $M \neq 0$ a f.g. R-module with $\mathrm{pdim}(M) < \infty$. Then

(i) $$\mathrm{pdim}(M) + \mathrm{depth}(M) = \mathrm{depth}(R).$$

<u>Proof</u>. Do induction on $d = \mathrm{pdim}(M)$. When $d = 0$, M is free, hence $\mathrm{depth}(M) = \mathrm{depth}(R)$ by fact 2.3.1. Let $d > 0$, let $m_1, \dots, m_k$ be generators of M forming a base of $M/\mathfrak{m}M$ and let $P = R^k \longrightarrow M$ be the surjection $(r_i)_{i=1}^k \longmapsto \sum_{i=1}^k r_i m_i$. We obtain a s.e.s

(+) $$0 \longrightarrow M' \xrightarrow{i} P \longrightarrow M \longrightarrow 0$$

such that (%) $i(M') \subseteq \mathfrak{m} \cdot P$ by our choice of the x_i,

and (&) $$\mathrm{pdim}(M') = \mathrm{pdim}(M) - 1$$

If $d = 1$ this means that M' is free, $M' = R^\ell$, and i is given by a $(k \times \ell)$-matrix with coefficients in $\mathfrak{m}$

(by (%)) hence $\mathrm{Ext}^j(k, M') \xrightarrow{\cong} \mathrm{Ext}^j(k, P)$ vanishes and by

$$\mathrm{Ext}^j(k, M') \xrightarrow{0} \mathrm{Ext}^j(k, P) \to \mathrm{Ext}^j(k, M) \to \mathrm{Ext}^{j+1}(k, M') \xrightarrow{0} \mathrm{Ext}^{j+1}(k, P)$$

we see

$$\mathrm{depth}(M) = \min(\mathrm{depth}(p), \mathrm{depth}(M') - 1)$$

But $M' \neq 0$ (as $d > 0$) and M' is free, hence $\mathrm{depth}(M') = \mathrm{depth}(p)$, and $\mathrm{depth}(M) = \mathrm{depth}(M') - 1$, establishing (i) & in this case $\mathrm{pdim}(M) = 1$.

Let now $d > 1$, and $c = \mathrm{depth}(M')$. By the induction assumption and (&) we have $(d-1) + c = \mathrm{depth}(R)$, hence $\mathrm{depth}(R) > c+1$. It follows that

$$0 = \mathrm{Ext}^i(k, P) \to \mathrm{Ext}^i(k, M) \xrightarrow{\cong} \mathrm{Ext}^{i+1}(k, M) \to \mathrm{Ext}^{i+1}(k, P)$$

when $i \le c$, hence $\mathrm{depth}(M) = c - 1$, showing (i) for M.

<u>Lemma 2.3.1</u>. For a f.g. R-module M, the following conditions are equivalent:

a) $\mathfrak{m} \notin \mathrm{Ass}(M)$

b) $\mathrm{depth}(M) > 0$

c) There is $x \in \mathfrak{m}$ s.t. $M \xrightarrow{x} M$ is injective.

<u>Proof</u>. a) $\Leftrightarrow$ b) follows from the characterization of depth from proposition 2.3.1 b).

b) $\Leftrightarrow$ c) Use

$$\{r \in R \mid M \xrightarrow{r} M \text{ not injective}\} = \bigcup_{\mathfrak{p} \in \mathrm{Ass}(M)} \mathfrak{p}$$

and prime avoidance.

<u>Lemma 2.3.2</u>. Let $x \in \mathfrak{m}$ be M-regular. Then

$$\mathrm{depth}(M) = \mathrm{depth}(M/xM) + 1.$$

<u>Proof</u>. From $0 \to M \xrightarrow{x} M \to M/xM \to 0$ we get

$$\mathrm{Ext}^i(k, M) \underset{*\,(\in \mathfrak{m})}{\xrightarrow{=0}} \mathrm{Ext}^i(k, M) \to \mathrm{Ext}(k, M/xM) \to \mathrm{Ext}^{i+1}(k, M) \to 0$$

showing the claim.

By induction we obtain:

<u>Corollary 2.3.1</u> If $(x_1, \dots, x_k)$ is an M-regular sequence of elements of $\mathfrak{m}$ then

$$\operatorname{depth}(M) = \operatorname{depth}\left(M/_{x_1M + \dots + x_kM} \right) + k.$$

<u>Proposition 2.3.2</u>. Let $M \neq 0$ be a f.g. R-module, then the following conditions are equivalent.

a) $\operatorname{depth}(M) \geq n$

b) M-regular sequences $(x_1, \dots, x_j) \in \mathfrak{m}^j$ for $0 \leq j \leq n$ can be extended to an M-regular sequence $(x_1, \dots, x_n) \in \mathfrak{m}^n$

c) There is an M-regular sequence $(x_1, \dots, x_n) \in \mathfrak{m}^n$?

<u>Proof</u>. Apply induction using the previous lemmas.

Recall that for completion $\wedge$ w.r.t. an Ideal $I \subseteq \mathfrak{m}$, we have

$$\operatorname{Ext}^i_{\hat{R}}(k, \hat{M}) = \widehat{\operatorname{Ext}^i_R(k, M)} = \operatorname{Ext}^i_R(k, M)$$

<u>Proposition 2.3.3</u>: If $\wedge$ denotes the completion w.r.t. $I \subseteq \mathfrak{m}$, then

$$\operatorname{depth}_{\hat{R}}(\hat{M}) = \operatorname{depth}_R(M).$$

<u>Proposition 2.3.4</u>: (F. Ischebeck) If $i < \operatorname{depth}(M) - \dim(T)$, then $\operatorname{Ext}^i(T, M) = 0$.

<u>Proof</u>: Do induction on $d = \dim(T)$. If $d = 0$, this is our definition of depth, see remark 2.3.1. Let $d \geq 1$. We have a filtration $0 = T_0 \subseteq \dots \subseteq T_k = T$ with $T_i/_{T_{i-1}} \cong R/_{\mathfrak{p}_i}$ and $\dim(\mathfrak{p}_i) \leq d$. Hence it is sufficient to ~~show~~ consider $T = R/\mathfrak{p}$, with $\dim(R/\mathfrak{p}) = d$. If $x \in \mathfrak{m} \setminus \mathfrak{p}$ then $0 \longrightarrow T \xrightarrow{x} T \longrightarrow T' = T/_{xT} \longrightarrow 0$, and $\dim(T') = \dim(T)$, hence by

$$\operatorname{Ext}^i(T, M) \xrightarrow{x} \operatorname{Ext}^i(T, M) \longrightarrow \operatorname{Ext}^{i+1}(T', M)$$

so $\operatorname{Ext}^i(T, M) \xrightarrow{x} \operatorname{Ext}^i(T, M)$ is surjective for $i+1 < \operatorname{depth}(M) - \dim(T)$ by induction, hence for $i < \operatorname{depth}(M) - \dim(T)$. By NAK $\operatorname{Ext}^i(T, M) = 0$ for such i.

Fact: Let R be Noetherian, M f.g.
Then the irreducible components of $\operatorname{supp} M = \{\mathfrak{p} \subset \operatorname{Spec} R \mid M_{\mathfrak{p}} \neq 0\}$ are the non-embedded primes of M (associated prime ideals not containing any other prime ideal associated to M.)

Proof: $M_{\mathfrak{p}} \neq 0 \iff \operatorname{Ass}(M_{\mathfrak{p}}) \neq \emptyset$
$$\iff \{\mathfrak{q} \in \operatorname{Ass}(M) \mid \mathfrak{q} \subseteq \mathfrak{p}\} \neq \emptyset$$
$$\iff \mathfrak{p} \in \bigcup_{\mathfrak{q} \in \operatorname{Ass}(M)} V(\mathfrak{q})$$

Corollary 2.3.2 i) If $\mathfrak{p} \in \operatorname{Ass}(M)$, then $\dim(R/\mathfrak{p}) \geq \operatorname{depth}(M)$
$$(\operatorname{Hom}_R(R/\mathfrak{p}, M) \neq 0 \text{ if } \mathfrak{p} \in \operatorname{Ass}(M))$$
ii) If $(x_i)_{i=1}^r \in \mathfrak{m}^r$ is an M-regular sequence and $\mathfrak{p} \in \operatorname{Ass}(M/(x_1M + \dots + x_rM))$, then
$$\dim(R/\mathfrak{p}) \geq \operatorname{depth}(M) - r$$
(as $\operatorname{depth}(M/\underline{x}M) = \operatorname{depth}(M) - r$ by corollary 2.3.1)

2.4 Cohen-Macaulay (CM) rings

Let R be a Noetherian local ring. We assume all R-modules to be f.g. unless otherwise stated.

Fact 2.4.1 $\dim(M) \geq \operatorname{depth}(M)$

Proof: We can use corollary 2.3.2 i) as
$$\dim(M) := \dim(\operatorname{Ass}(M)) = \max_{\mathfrak{p} \in \operatorname{Ass}(M)} \dim(R/\mathfrak{p})$$
by the fact recalled above.

In particular
$$\operatorname{depth}(R) \leq \dim(R)$$

Definition 2.4.1. A Noetherian local ring R is Cohen-Macaulay if
$$\operatorname{depth}(R) = \dim(R)$$

Remark 2.4.1 Recall that I is an ideal of definition if $\sqrt{I} = \mathfrak{m}$. A parameter sequence of R is a sequence of $\dim(R)$ elements of R generating an ideal of definition.
Parameter sequences always exist by by Hilbert-Samuel polynomial theory.
By the results of the previous subsection, the length of any regular sequence of elements of $\mathfrak{m}$ is $\leq \operatorname{depth}(R) \leq \dim(R)$
Thus R is CM iff a regular parameter sequence may be found.
Conversely, if $d = \dim(R)$ and $(x_1, \dots, x_d)$ is regular and

$\mathfrak{p} \in V(\langle x_1 \dots x_d \rangle_R)$. Then the x_i form a regular sequence of elements of the maximal ideal of $R_\mathfrak{p}$, hence $\dim(R_\mathfrak{p}) \geq \dim(R)$, hence $\mathfrak{p} = \mathfrak{m}$, showing $V(\langle x_1 \dots x_d \rangle_R) = \{\mathfrak{m}\}$ and thus

$$\sqrt{(\underline{x})_R} = \mathfrak{m}$$

and thus $\underline{x} = (x_1 \dots x_d)$ is a parameter sequence.

Theorem 3: If R is CM and $\mathfrak{p} \in \operatorname{Spec} R$, then $R_\mathfrak{p}$ is CM and

$$\dim(R_\mathfrak{p}) = \dim(R) - \dim(R/\mathfrak{p})$$

Proof: We have

$$\operatorname{Ext}^p_{R_\mathfrak{p}}(k(\mathfrak{p}), R_\mathfrak{p}) = \operatorname{Ext}^p_R(R/\mathfrak{p}, R)_\mathfrak{p} = 0$$

when $p < \operatorname{depth}(R) - \dim(R/\mathfrak{p})$.

As R is CM we have

$$\dim(R_\mathfrak{p}) \geq \operatorname{depth}(R_\mathfrak{p}) \geq \operatorname{depth}(R) - \dim(R/\mathfrak{p}) \geq \dim(R) - \dim(R/\mathfrak{p})$$

$$\geq \dim(R_\mathfrak{p}) = \operatorname{ht}(\mathfrak{p}),$$

hence we have equality and all assertions follow.

Corollary 2.4.1. Local CM rings are catenary and all irreducible components of Spec R are of dimension $\dim(R)$.

Definition 2.4.2. a) A Noetherian ring R is a Cohen Macaulay ring if $R_\mathfrak{p}$ is CM for all $\mathfrak{p} \in \operatorname{Spec} R$, or equivalently $R_\mathfrak{m}$ is CM for all $\mathfrak{m} \in \text{m-Spec}(R)$.

A locally Noetherian prescheme is CM iff all $\mathcal{O}_{X,x}$ are CM $\iff$ $\mathcal{O}_X(U)$ is CM for all open affine U $\overset{(X\ q.c.)}{\iff}$ $\mathcal{O}_{X,x}$ is CM when x is closed.

b) (Serre) A locally Noetherian prescheme is S_k if

$$\operatorname{depth}(\mathcal{O}_{X,x}) \geq \min(k, \dim(\mathcal{O}_{X,x}))$$

A loc. Noeth. prescheme is R_k if $\mathcal{O}_{X,x}$ is regular when $\dim(\mathcal{O}_{X,x}) \leq k$

Remark a) If $(x_i)_{i=1}^d$ are generators of the maximal ideal of a d-dimensional regular local ring, then they form a regular sequence, as we have seen early in 2.2.

Hence regular rings are CM.

b) Serre: X is reduced iff it is R_0 and S_1
X is normal iff it is R_1 and S_2

Proposition 2.4.1: If R is CM then $\hat{R}$ is CM.
Actually $\operatorname{depth}(R) = \operatorname{depth}(\hat{R})$.

Proof: $\mathrm{Ext}^p_R(k, R) = \widehat{\mathrm{Ext}^p_R(k, R)} = \mathrm{Ext}^p_{\hat{R}}(k, \hat{R})$.

Proposition 2.4.2: If R is CM then so are $R[T_1, \dots, T_n]$ and $R[[T_1, \dots, T_n]]$.

Corollary 2.4.2: CM rings are universally catenary.

Proof proposition: As $R[[T_1, \dots, T_n]] = \widehat{(R[T_1, \dots, T_n])}$ (w.r.t. $\langle T_1, \dots, T_n \rangle_{R[T_1, \dots, T_n]}$) it is sufficient to show the first assertion and by induction to show that $R[T]$ is CM.
Let $\mathfrak{p} \in \operatorname{Spec} R[T]$. W.l.o.g. let R be local and $\mathfrak{m} = \mathfrak{p} \cap R$ be the maximal ideal of R.
If $\mathfrak{p} = \mathfrak{m}[T]$, then any regular sequence of $\dim(R)$ elements of R is an $R[T]$ regular sequence of elements of $\mathfrak{p}$ ($R[T]$?). But $\operatorname{ht}(\mathfrak{p}) = \dim(R)$ in this case as $\dim(R[T]) \leq \dim(R) + 1$, by Hilbert polynomial theory.
If $\mathfrak{p} \supsetneq \mathfrak{m}[T]$, then $\mathfrak{p} = \langle \mathfrak{m}, P \rangle$, where $P \in R[T]$ is a normed polynomial s.t. P mod $\mathfrak{m}$ is a minimal polynomial of T mod $\mathfrak{p}$ over $k = R/\mathfrak{m}$. Let $d = \dim(R)$, and $(x_1, \dots, x_d) \in \mathfrak{m}^d$ an R-regular sequence, then $(x_1, \dots, x_d, P)$ is $R[T]$-regular, since $(x_1, \dots, x_d)$ clearly is and $R[T]/\sum_{i=1}^d x_i R[T] = \left(R/\sum_{i=1}^d x_i R\right)[T]$ is of the form $\Lambda[T]$ where Λ is a finite dimensional R-algebra. Hence $\Lambda[T]$ has a finite filtration with filtration quotients $\cong k[T]$, hence P is injective on it.
But
$$\sqrt{\langle x_1, \dots, x_d, P \rangle_{R[T]}} = \sqrt{\langle \underline{x} \rangle_R R[T] + P \cdot R[T]} = \sqrt{\mathfrak{m}[T] + P \cdot R[T]} = \mathfrak{p}.$$

Corollary 2.4.3 (of corollary 2.3.2)
If R is CM it is <u>unmixed</u> (i.e. has no embedded associated prime ideals) and all irreducible components of $\operatorname{Spec} R$ have the same dimension.

2.5 Gorenstein rings and locally complete intersection maps

Theorem: For a Noetherian local ring R, t.f.a.e.

a) $\mathrm{idim}_R(R) < \infty$

b) $\mathrm{idim}_R(R) = \dim R$

c) $\mathrm{Ext}^p_R(k, R)$ vanishes for some p

d) $\mathrm{Ext}^p_R(k, R)$ vanishes for $p > \dim(R)$

e) $\mathrm{Ext}^d_R(k, R)$ is one-dimensional

f) R has a parameter ideal which is irreducible in the sense of Laskau (i.e. not the intersection of two larger ideals)

g) Any parameter ideal is irreducible in the sense of Laskau.

Definition 2.5.1 Such rings are called Gorenstein

Example a) Regular local rings

b) $\mathbb{Z}/p^n$ or $R/\mathfrak{p}^n$ when R is a PID (Dedekind domain) and $\mathfrak{p} \neq 0$ a prime ideal.

c) $k[X,Y]/\langle X^{2018}, Y^{2012}\rangle$ is Gorenstein (Laskau-irreducibility criterion as any non-zero ideal must contain $X^{2017}Y^{2017}$)

d) $k[X,Y]/\langle X^2, XY, Y^2\rangle$ is not Gorenstein (but it is CM)

Theorem: For a local Noetherian ring R, t.f.a.e.

a) There is some parameter sequence $\underline{x}$ s.t.

$$\dim(H(\cdots)) = \dim(m/m^2) = \dim(R)$$

b) This holds for all parameter sequences.

When $R \cong S/I$ for some regular S, this is equivalent to

c) There is a regular S and an ideal $I \subseteq S$ generated by a regular sequence s.t. $R \cong S/I$

d) Wherever $R \cong S/I$ with S regular, I is generated by a regular sequence.

Definition 2.5.2. Such loc. Noeth rings are called local complete intersection rings. (lci)

Remark: Regularity $\underset{\not\Leftarrow}{\Rightarrow}$ lci $\underset{\not\Leftarrow}{\Rightarrow}$ Gorenstein $\underset{\not\Leftarrow}{\Rightarrow}$ CM $\underset{\not\Leftarrow}{\Rightarrow}$ universally catenary